Module
6

Applied Mathematics 30

PATTERNS

Learning Technologies Branch

Alberta LEARNING

Module 6

Applied Mathematics 30

PATTERNS

Learning Technologies Branch

Alberta LEARNING

Applied Mathematics 30
Module 6: Patterns
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-2297-8

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Home Instructors	
General Public	
Other	

You may find the following Internet sites useful:

- Alberta Learning, **http://www.learning.gov.ab.ca**
- Learning Technologies Branch, **http://www.learning.gov.ab.ca/ltb**
- Learning Resources Centre, **http://www.lrc.learning.gov.ab.ca**

The use of the Internet is optional. Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

Welcome

Applied Mathematics 30

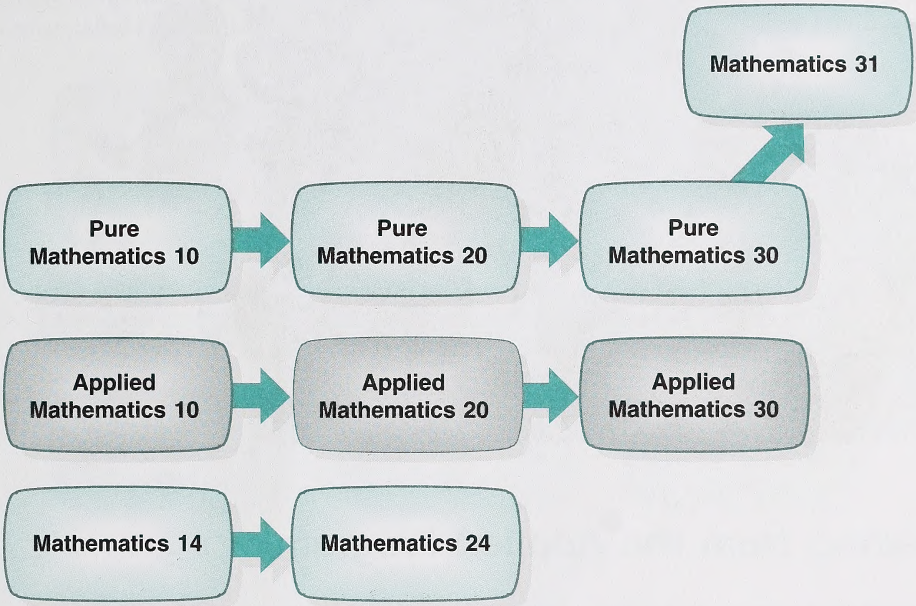
Welcome to Module 6.
We hope you'll enjoy your study of Patterns.

Applied Mathematics 30 contains seven modules and a final test. Work through the modules in the order given, since several concepts build on each other as you progress through the course.

Contents

Introduction to Applied Mathematics 30

Applied Mathematics 30 is the third course in the Applied Mathematics 10–20–30 program of studies. Another program of studies is Pure Mathematics 10–20–30; students who complete Pure Mathematics 30 often choose to take Mathematics 31. A third program of studies is Mathematics 14–24.



Each mathematics program is designed for students with different mathematical strengths and interests.

- Pure Mathematics 10–20–30 is intended for students who are strong in algebra and mathematical theory.
- Applied Mathematics 10–20–30 is better suited to students who prefer to solve problems using numerical reasoning or geometry.
- Mathematics 14–24 is a general mathematics program for high school students who have experienced difficulties in previous mathematics courses.

Each sequence of courses is designed for students with different career plans. For example, Pure Mathematics 30 is a prerequisite for admission to many university programs. Many colleges and technical institutes, however, will admit students who have successfully completed Applied Mathematics 30.

You may find it helpful to read any of the documents under the heading "New Senior High School Mathematics Update/Post-Secondary Studies Update" at the following Internet site:

http://www.learning.gov.ab.ca/k_12/curriculum/bySubject/math

Before enrolling in Applied Mathematics 30, it is recommended that you talk with a school counsellor about your career plans.

Transferring from the Applied Program

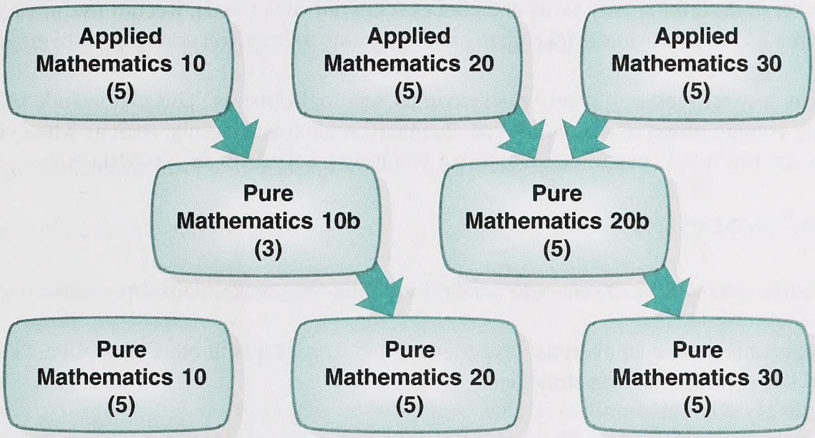
You should be aware that the applied and pure mathematics courses do have some topics in common; other topics are independent.

The following table shows some common and independent topics.

Applied Topics	Common Topics	Pure Topics
• linear programming • data tables and trends • design and layout • metric and imperial measure • data presentation • vectors and matrices • periodic, fractal, and recursive patterns • financial decision making • costing and design problems	• spreadsheets • line segments and linear graphs • scaling • triangles • financial mathematics • quadratic functions • circle geometry • the bell curve	• irrational numbers • exponents • polynomial and rational expressions • mathematical expectations • growth patterns • linear and non-linear systems • operations on functions • mathematical reasoning • exponential and logarithmic functions • conics • combinations • trigonometric functions

If you want to transfer from the Applied Mathematics 10–20–30 sequence to the Pure Mathematics 10–20–30 sequence at a future time, you won't have to repeat the topics that are common to pure mathematics and applied mathematics.

If you decide to transfer to Pure Mathematics 20 after successfully completing Applied Mathematics 10, you may have to take a three-credit course called Pure Mathematics 10b. If you decide to transfer to Pure Mathematics 30 after successfully completing Applied Mathematics 20 or Applied Mathematics 30, you may have to take a five-credit course called Pure Mathematics 20b. The two bridging courses are shown in the following diagram.

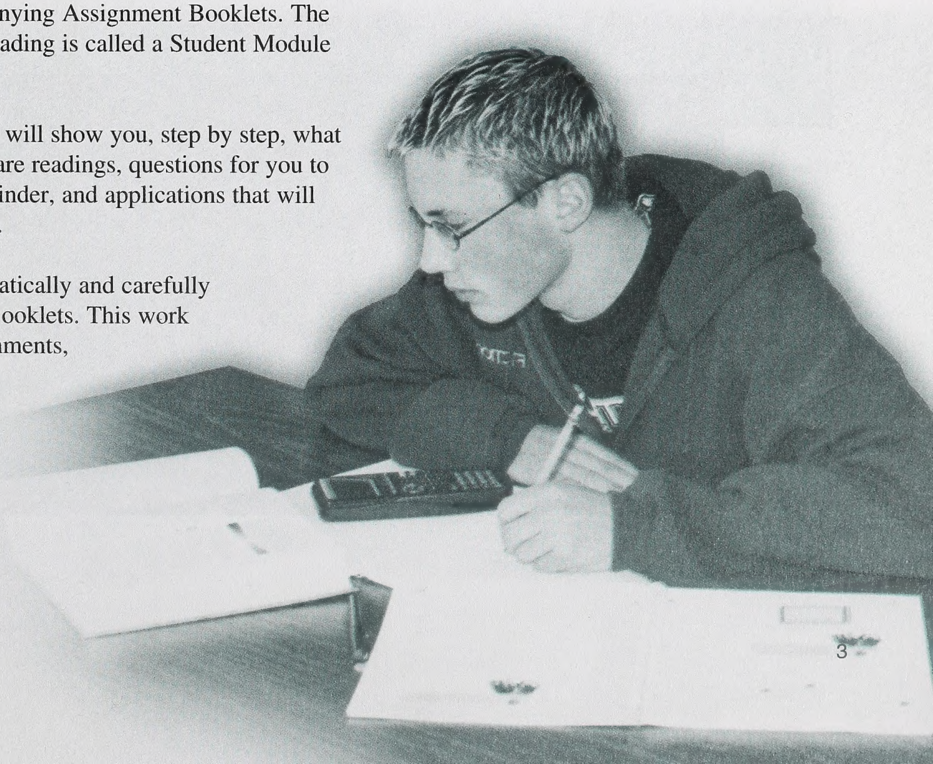


Strategies for Completing Applied Mathematics 30

For each module in Applied Mathematics 30, there is a Student Module Booklet and accompanying Assignment Booklets. The document you are presently reading is called a Student Module Booklet.

Each Student Module Booklet will show you, step by step, what to do and how to do it. There are readings, questions for you to answer in your mathematics binder, and applications that will give you hands-on experience.

It is important to work systematically and carefully through the Student Module Booklets. This work will prepare you for the assignments, projects, and final test.



Following are some suggestions for organizing your mathematics binder:

- Keep a section of your binder to record your responses to the questions in the Student Module Booklet. Also store your marked assignments here.

- Keep a section of your binder for work in progress on your projects. Keep your research notes, plans, rough drafts, and so on.

- Keep a section of your binder to record new skills and concepts, as well as important results and formulas. Get in the habit of describing new skills and concepts in your own words. Record useful ways to help you remember what a concept means. Make charts and diagrams to help you connect mathematical ideas.

- Keep a section of your binder to record mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you grasped a difficult concept (an "aha!" experience), or when you used a calculator or spreadsheet in a new way.

Mathematical Processes

Throughout this course, you will be expected to perform the following mathematical processes:

- Connect mathematical ideas to everyday experiences and to concepts in other disciplines.
- Develop and use problem-solving strategies.
- Reason and justify your answers.
- Communicate mathematical ideas.
- Select and use appropriate technologies to solve problems.
- Develop and use estimation and mental-math strategies.
- Use visualization to assist in processing information, making connections, and solving problems.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Applied Mathematics 30 to be your study partner. You will find that having a friend to discuss mathematical ideas with will make your studying more enjoyable.

Resources You Will Need

In addition to the course materials for Applied Mathematics 30, you will need the following resources:

- the *Addison-Wesley Applied Mathematics 12 Source Book*, Western Canadian Edition, published by Addison Wesley Longman Ltd. (2002)
- the *Addison-Wesley Applied Mathematics 12 Project Book*, Western Canadian Edition, published by Addison Wesley Longman Ltd. (2002)
- a binder, lined loose-leaf paper, graph paper, dividers, pencils, eraser
- metric and imperial measuring devices, such as a ruler, yardstick, metre-stick, and tape measure
- a mathematical instrument set (compass, protractor, and triangles)
- a computer with a spreadsheet program

  Note: Two popular spreadsheet programs are *ClarisWorks*™ and Microsoft® *Excel*.

- a graphing calculator

  Note: Where it is applicable, the examples in this course and the textbook show the TI-83 calculator; however, all of the graphing calculators in the following chart are approved for use on tests.

Texas Instruments	Sharp	Casio	Hewlett-Packard
TI-83	EL-9600C	Algebra FX 2.0	HP 39g†
TI-83 Plus	EL-9600*	CFX-9850 GA-Plus*	
TI-86	EL-9200*	CFX-9850 G*	
TI-89	EL-9300*	CFX-9800 G*	
TI-92*		FX-9700 series*	
TI-92 Plus			

*no longer commercially available
† The HP 39g calculator will remain on the approved list for the 2001–2003 school years and will then be deleted from the approved list.

If you intend to use the TI-83 or TI-83 Plus graphing calculator, it is recommended that you obtain the video program *The TI-83 Graphing Calculator Video Tutor*.

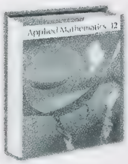
Many of the resources you will need may be purchased locally or from the Learning Resources Centre (LRC). Following is the LRC website:

http://www.lrc.learning.gov.ab.ca

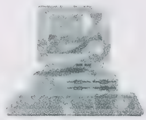
You may wish to discuss the availability of resources with your teacher, as your school division may have a loan policy.

Visual Cues

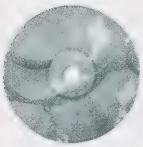
You will find many visual cues in this course. Colour is used to highlight terms that are defined in the Glossary of the Appendix of each Student Module Booklet. You will also find several icons in the margins. Read the following explanations to discover what the various icons prompt you to do.



Refer to the textbook or the Project Book.

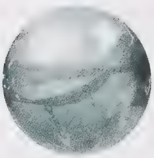


Work with a computer.



Refer to the Applied Mathematics 30 CD.

Contact your teacher for additional information.



Explore the Internet.

Complete specified questions in the Assignment Booklet.

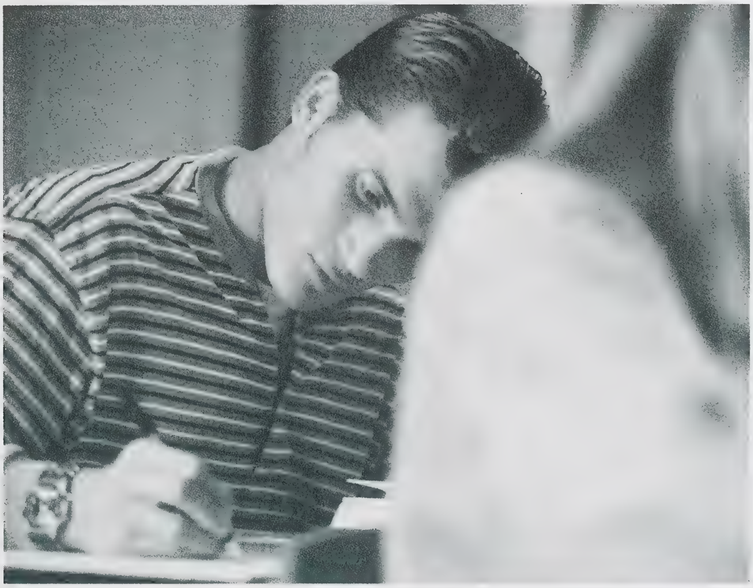
Remember: Any Internet website address given in this module is subject to change.

Where Can I Obtain Diploma Examination Information?

Alberta students will write a diploma examination at the end of the course. Alberta Learning provides several documents to help students prepare for this examination. These documents are found under the heading "Diploma Examinations" at the following Alberta Learning website:

http://www.learning.gov.ab.ca/k_12/testing

Information like course expectations, the makeup of the diploma examination, keyed copies of previous examinations, preparation guides, and calculator policies are available to students at this site.



Each year, in February and September, Alberta Learning provides teachers with information on a **student project**, which teachers **may** use as part of your overall assessment. Information to students will also be posted on the Alberta Learning website. Check with your teacher to determine what you will be expected to do. Be aware that one of the diploma examination's written-response questions will deal with elements of this project and is worth 10% of your diploma examination mark.

You should take advantage of the many sources of information about Applied Mathematics 30. Your success depends on your understanding of course expectations and evaluation procedures. Work closely with your teacher and do not hesitate to ask questions.

Remember, take the initiative to find out all you can about Applied Mathematics 30.

MODULE OVERVIEW

Have you had a cold or cough recently? Have you had to take cough medicine to help relieve your cough? The directions on a cough-medicine container may tell you to take 20 mL every 4 h. While your body eliminates some of the medication over time, a certain amount is required to maintain the effectiveness of the medicine. The data representing the amount of medication present in your bloodstream over a period of time forms a pattern.

In this module, you will study various types of patterns, including those that represent the amount of medication present in the bloodstream when taking medication. You will determine the pattern in a sequence, identify arithmetic and geometric sequences, and use technology to generate a sequence. You will also model problems using sequences and graph sequential data. You will then investigate the patterns involved in fractals and calculate the perimeter, area, and volume of a fractal.

If you are required to complete the module project, you will use patterns and sequences to analyse how much caffeine remains in your body over a period of time and how it affects your body.

Assessment

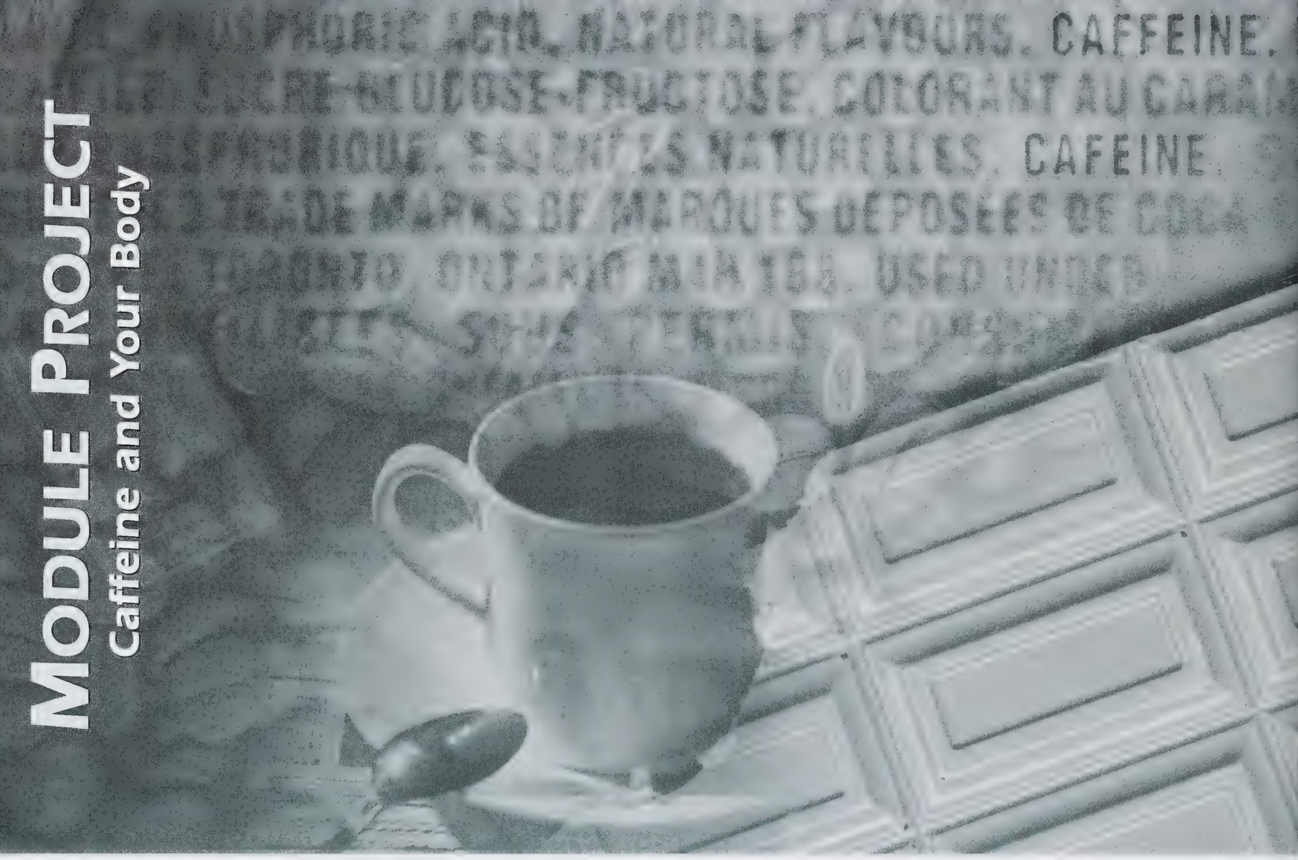
Accompanying this Student Module Booklet are two Assignment Booklets. Your grading in this module will be based upon the assignments you submit for assessment. The mark distribution is as follows:

Assignment Booklet 6A	
Activities 1 and 2 Assignment	**75 marks**
Assignment Booklet 6B	
Activities 3 and 4 Assignment	**40 marks**
Module Review Assignment	**40 marks**
Module Project	**40 marks**
TOTAL	**195 marks**

Remember that Activities 1 to 4 in this Student Module Booklet will prepare you for completing the module project and the module assignments. You should work through these activities carefully and compare your answers with the suggested answers provided in the Appendix.

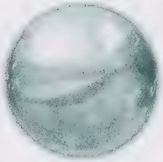
The Module Review provides a review of the module and an enrichment activity. You may choose to do some or all of the questions in the Module Review. Again, you should compare your answers with the suggested answers provided in the Appendix.

Module Project
Caffeine and Your Body



Beginning the Project

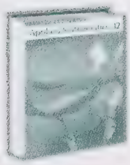
Your teacher may not require you to complete all the projects provided in this Applied Mathematics 30 course. Contact your teacher and check whether you need to complete the module project, Caffeine and Your Body, as part of your assessment.



Most Canadians feel tremendous pride when one of Canada's athletes wins a gold medal. We also feel tremendous shame when one of our athletes is caught cheating by using a banned, performance-enhancing substance. You can read more about substances banned for use by Olympic athletes at the following website:

http://www.wada-ama.org

One of the banned substances might surprise you, since it is in many foods and drinks people consume everyday. That substance is caffeine, and it will be the subject of the module project.



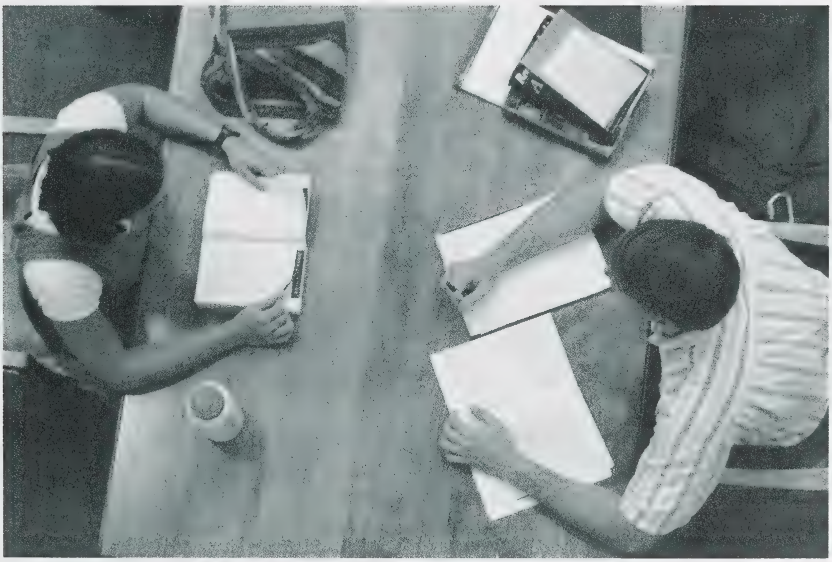
Turn to page 258 of the textbook and read "Caffeine and Your Body." Answer the questions posed, and store them in the project section of your mathematics binder.

To begin your research, refer to the Internet site given at the bottom of page 258 of the textbook. This website provides several links to other sites you may find helpful.

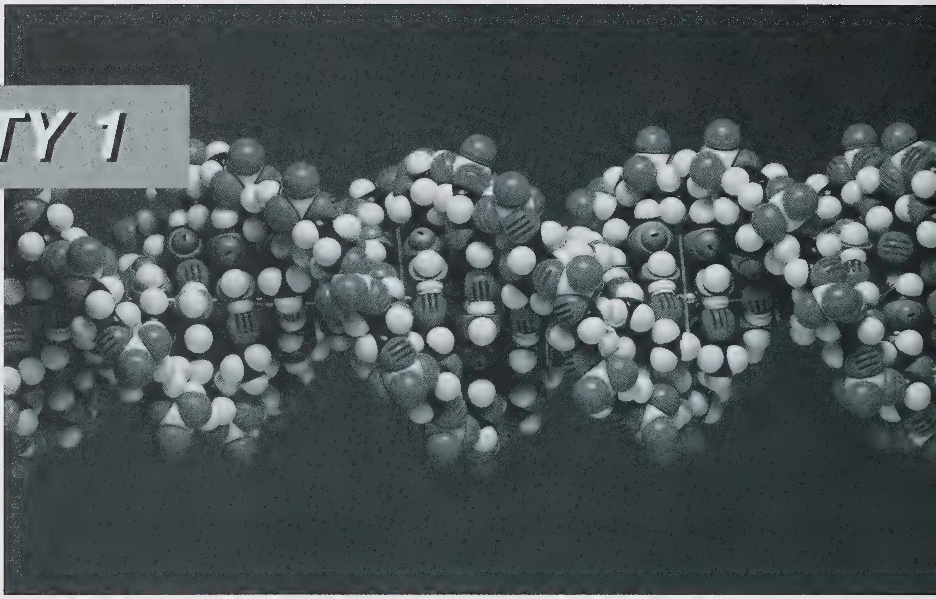
Note: The topics are listed under the heading Caffeine and Your Body, not The Effects of Caffeine.

As you work through Activities 1 through 4, continue to research ideas that will help you complete the project. You will find the concepts and exercises presented in this module, such as half-life and geometric sequences, quite useful. You should be working on this part of the project while you work though Activities 1 to 4.

Working through Activities 1 to 4 will help you gain the skills and concepts needed to complete this module project. You will be given more direction on how to complete this project later in this module. In the meantime, feel free to discuss your project with your study partner or a family member. Remember, the work on the project you submit must be your own.



ACTIVITY 1



Sequences

The deoxyribonucleic acid (DNA) helix is at the centre of scientists' understanding of what makes up all living organisms. Did you know that sequences of four nucleotides spell out the genetic material found in the nucleus of cells? Every person except for those of you who have an identical twin, has a unique set of genetic material built up of these proteins. Identifying DNA sequences is providing new ways to attack disease and to develop therapies that treat genetically based illnesses.

In this activity you will work with mathematical sequences. You, too, will have to identify sequences and their properties and use them to solve problems.

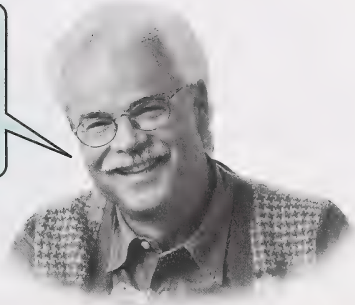
Turn to page 260 of the textbook and read the introductory paragraphs of Tutorial 6.1, "Sequences." Pay particular attention to the three kinds of mathematical sequences mentioned:

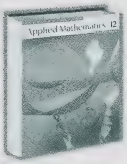
- **arithmetic sequences**
- **geometric sequences**
- **recursively generated sequences**

1. According to horticulturists at the University of Saskatchewan, mature potato plants have 5 to 10 fully developed potatoes on average. How many potatoes might one potato plant have as its descendants after five growing seasons?

Compare your response with the suggested answer in the Appendix, Activity 1, page 45.

From exercise 1, you can see that numbers forming a geometric sequence increase very quickly with each step. You will see later in the module that the change in a geometric sequence is exponential.





2. Turn to page 261 of the textbook and complete exercises 1 to 8 of "Investigation 1: The Growth of Bacteria."

Compare your responses with the suggested answers in the Appendix, Activity 1, pages 45–47.

Example

For each sequence given, determine whether it is arithmetic, geometric, or neither.

Sequence A: 45, 30, 15, 0, $-15, \ldots$
Sequence B: 324, -108, 36, -12, 4, ...
Sequence C: 2, 7, 18, 28, 182, ...
Sequence D: 4.85, 4.16, 3.47, 2.77, 2.08, ...

Solution

To test if a sequence is arithmetic, you must find the differences between succeeding terms. Remember, if these differences are equal, the sequence is arithmetic.

Sequence A

Term	45	30	15	0	−15
Difference	$30-45=-15$	$15-30=-15$	$0-15=-15$	$-15-0=-15$	

Sequence B

Term	324	−108	36	−12	4
Difference	-432 $(-108-324)$	144 $[36-(-108)]$	-48 $(-12-36)$	16 $[4-(-12)]$	

Sequence C

Term	2	7	18	28	182
Difference	5 $(7-2)$	11 $(18-7)$	10 $(28-18)$	154 $(182-28)$	

Sequence D

Term	4.85	4.16	3.47	2.77	2.08
Difference	-0.59 $(4.16-4.85)$	-0.69 $(3.47-4.16)$	-0.70 $(2.77-3.47)$	-0.69 $(2.08-2.77)$	

Sequence A is an arithmetic sequence. Its differences are all the same, -15.

To test if a sequence is geometric, you must find the ratios of succeeding terms. If these ratios are equal, the sequence is geometric.

Sequence B

Term	324	−108	36	−12	4
Ratio	$\frac{-108}{324}=-\frac{1}{3}$	$\frac{36}{-108}=-\frac{1}{3}$	$\frac{-12}{36}=-\frac{1}{3}$	$\frac{4}{-12}=-\frac{1}{3}$	

Sequence C

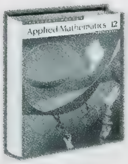
Term	2	7	18	28	182
Ratio	$\frac{7}{2}=3.5$	$\frac{18}{7}\doteq 2.6$	$\frac{28}{18}\doteq 1.6$	$\frac{182}{28}=6.5$	

Sequence D

Term	4.85	4.16	3.47	2.77	2.08
Ratio	$\frac{4.16}{4.85}\doteq 0.86$	$\frac{3.47}{4.16}\doteq 0.83$	$\frac{2.77}{3.47}\doteq 0.80$	$\frac{2.08}{2.77}\doteq 0.75$	

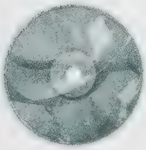
Sequence B is a geometric sequence. Its ratios are all the same, $-\frac{1}{3}$.

Sequences C and D are neither arithmetic nor geometric. They have neither a common difference nor a common ratio.



Turn to pages 261 to 263 of the textbook and complete the steps outlined in exercises 1 to 6 of "Investigation 2: A Bouncing Ball." If you do not have access to a CBR (or CBL) with a motion detector to collect data, open the explorer *Ball Bounce 1* on the Applied Mathematics 30 CD. **Note:** Once the program starts, a trial bounce automatically begins. Feel free to use the initial data collected, or use the data produced by releasing the ball at any height you wish.

The program *Ball Bounce 1* stops recording bounce heights before the ball stops bouncing. This becomes more obvious as you change the scale from 1 to 45. You will find it gives less than realistic results for the larger scale values. The ball will probably reach terminal velocity from heights less than the maximum the program uses, so the bounces will not have the nice relationship that the program provides. You may wonder if a soccer ball moving at terminal velocity will bounce or go splat when it hits the ground.

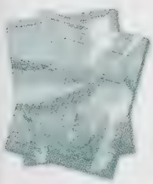


3. Complete exercises 7, 8, and 9 of "Investigation 2: A Bouncing Ball" on pages 262 and 263 of the textbook. **Note:** If you do not have access to a CBR with a motion detector or if you are unable to use the explorer *Ball Bounce 1*, use the following information.

Bounce	1	2	3	4	5	6	7
Height (m)	19.6	14.4	10.0	7.2	4.9	3.6	2.5

4. Answer questions 2, 3, and 4 of "Discussing the Ideas" on pages 263 and 264 of the textbook.

Compare your responses with the suggested answers in the Appendix, Activity 1, pages 48–49.



Turn to page 1 of Assignment Booklet 6A and answer questions 1 and 2.

Turn to page 263 of the textbook and work through "Example 1: Use Technology to Determine a Term of a Sequence." You will practise using the Sequence mode on your graphing calculator to generate recursive sequences.

If you are using the TI-83 graphing calculator and need information on how to put your calculator in Sequence mode and/or to set up appropriate table settings, refer to "Utility 33: Using the TI-83 to Generate and Graph a Sequence" on pages 363 and 364 of the textbook.

In the preceding example, the common factor in the formula for *u(n)* was 2. You can also use the graphing calculator to generate a sequence with a common difference (a common term that is added or subtracted).

Example

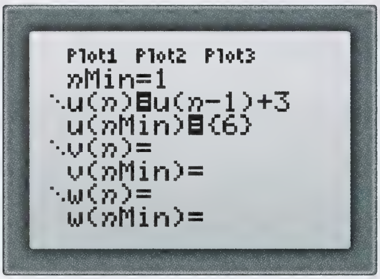
Use your graphing calculator to generate the following sequence; then find the 11th term.

6, 9, 12, 15, …

Solution

The common difference between successive terms is $+3$.

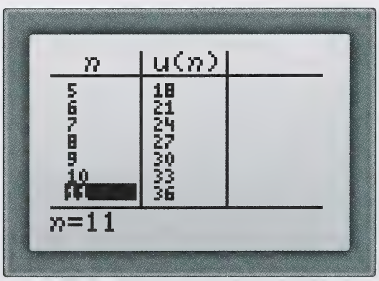
Press Y= and enter the following settings.



Because each term is generated by adding 3 to the preceding term, use the formula $u(n) = u(n-1) + 3$.

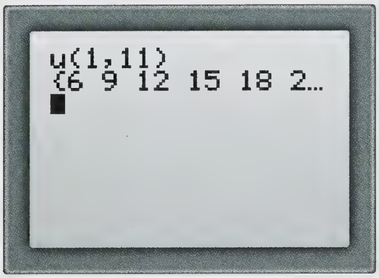
Method 1: Using the Table Values

Press 2nd [TABLE], and scroll down to $n = 11$.



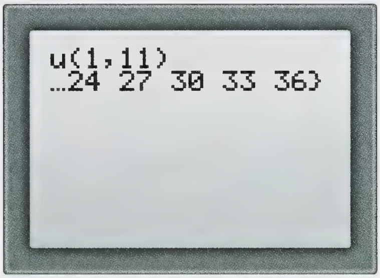
The 11th term is 36.

Method 2: Using the Sequence Command



$u(1, 11)$ is the sequence of terms 1 to 11.

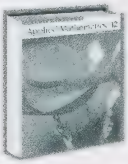
To find the 11th term, count the terms and use the right-arrow key to move the cursor until it reaches the 11th term. In this case, the 11th term is the last term.



The 11th term is 36.

Again, you may wish to refer to Utility 33 on pages 363 and 364 of the textbook for additional information regarding generating a sequence on your T1-83 graphing calculator.

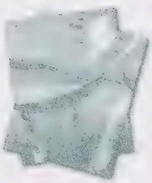
Mathematical sequences show up in places that might surprise you. The perspective in the scenes like those shown here have sequences involved. If you were to measure the heights of the posts in the picture on the left or the lengths of the train cars in the picture on the right, you could determine a sequence.



5. Turn to pages 264 and 265 of the textbook and answer the following.

   a. exercises 1.a., 1.c., 1.e., 3, 4, and 7 of "Exercises: Checking Your Skills"
   b. exercise 10 of "Exercises: Extending Your Thinking"

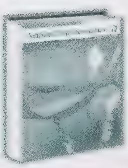
Compare your responses with the suggested answers in the Appendix, Activity 1, pages 50–53.



Turn to pages 2 and 3 of Assignment Booklet 6A and answer questions 3 and 4.

Looking Back

In this activity you studied the characteristics of three types of mathematical sequences: arithmetic sequences, geometric sequences, and recursively generated sequences. You also used technology to generate and use arithmetic and geometric sequences to solve real-world problems.



6. Turn to page 265 of the textbook and answer "Communicating the Ideas."

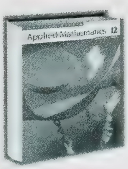
Compare your response with the suggested answer in the Appendix, Activity 1, pages 53–54.

ACTIVITY 2

Modelling Using Sequences

When you were younger, did your parents keep track of your height on the back of a door or on the wall in an inconspicuous spot? Did you see how much taller you had become since the last time you were measured? A sequence like 87, 90, 103, 115, 123, 128, 133, 139, 143, 149, 156, … or a sequence like 85, 93, 101, 108, 114, 122, 127, 133, 137, 144, 151, 157, … could have represented your growth rate. When you were younger, your height would increase and a new number would appear in the sequence. The human body, how it grows and how it functions, provides a large number of activities that generate sequences. Some are as obvious as growing taller as we progress from infancy to adulthood; others, such as cell division, are so hidden away that it takes sensitive scientific equipment to find them.

In this activity you will look at how sequences can be used to model real-world occurrences.

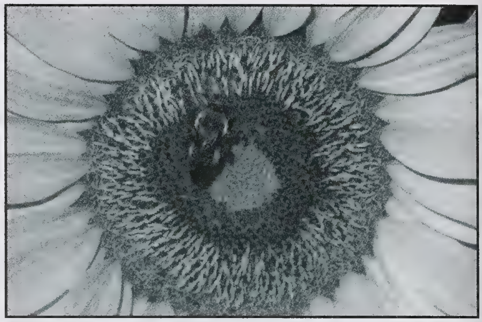


Turn to page 266 of the textbook and read the introductory paragraphs of Tutorial 6.2, “Modelling Using Sequences.”

1. Complete exercises 1 to 5 of “Investigation 1: Graph a Sequence Using a Spreadsheet Program” on pages 266 and 267 of the textbook. **Note:** It is not necessary to work with a partner in Investigation 1.

Compare your responses with the suggested answers in the Appendix, Activity 2, pages 54–56.

Nature provides a huge number of interesting designs, shapes, and sequences. Some of them are shown in the images that follow.



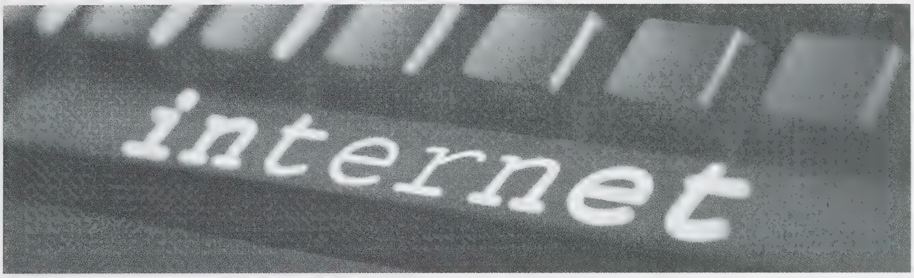
The seashell's spiral shape and the packing of seeds in a sunflower are both examples of the Golden Mean (sometimes call the Golden Ratio). Each step is related to the ratio $\frac{\sqrt{5}-1}{2} \doteq 1.618\ 033\ 988\ 7$.

Builders have also used this same ratio for more than 2400 years. In fact, the Parthenon in Athens, Greece, was designed using the Golden Mean for many of its shapes circa 400 B.C.

Did you know that a copy of the Parthenon was built in Nashville, Tennessee, in the late 1920s that matches it to within one eighth of an inch?

For more information on the Golden Mean (or Golden Ratio), use one of the Internet's search engines to view some relevant websites. You may start by visiting the following:

http://www.vashti.net/mceinc/golden.html



Example

Use the infinite sequence given to answer the questions that follow.

1, …, 46 368, 75 025, 121 393, 196 418, 317 811, 514 229, …

a. Is the sequence arithmetic, geometric, or neither?
b. Predict the next 4 terms of the sequence.
c. Predict the 3 terms of the sequence preceding 46 368.

Solution

a. Calculate the differences between consecutive terms of the sequence, and calculate the ratios of consecutive terms of the sequence. The results are shown in the following table.

Term	Difference	Ratio
46 368		
75 025	28 657	1.618 033 989
121 393	46 368	1.618 033 989
196 418	75 025	1.618 033 989
317 811	121 393	1.618 033 989
514 229	196 418	1.618 033 989

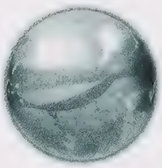
The differences between consecutive terms are not constant, so the sequence is not arithmetic. The ratios, however, are the same. Therefore, this sequence is geometric.

b. The next four terms can be predicted by multiplying the preceding term by 1.618 033 989. The next four terms are, therefore, 832 040.000 1, 1 346 269, 2 178 309.001, and 3 524 578.002.

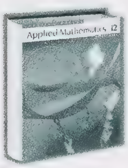
Since the original sequence has only whole-number values, the decimal parts suggest some small error in these new values. Is this enough to say the sequence is not geometric? If you look closely at the differences in the previous table, you will notice that the sequence itself shows up in the Difference column a couple of rows farther down. For doing the prediction requested, the sequence can be considered geometric. As you may remember from earlier mathematics courses, making predictions doesn't always give exact, correct values. The idea is to have a good sense of what the needed values might be. In this case, you might want to "fix" the numbers by taking the closest whole-number values.

Therefore, the next four terms are 832 040, 1 346 269, 2 178 309, and 3 524 578.

c. The three terms preceding 46 368 can be predicted by dividing the successive term by 1.618 033 989. The three terms preceding 46 368 are, therefore, 28 656, 17 711, and 10 945.



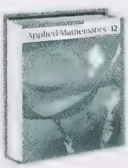
The sequence in the preceding example is actually a famous sequence known as the Fibonacci Numbers. Use one of the Internet's search engines to research the Fibonacci sequence, and find many more interesting places in the real world in which it appears.



Turn to page 267 of the textbook and read the information between Investigation 1 and Investigation 2.

2. Complete exercises 1 to 10 of "Investigation 2: The Metabolism of Insulin" on pages 267 and 268 of the textbook.

Compare your responses with the suggested answers in the Appendix, Activity 2, pages 57–58.



Turn to page 268 of the textbook and read the information at the bottom of the page. Then work through "Example: Modelling the Effect of Repeated Intake of a Substance" on page 269. **Note:** The two statements at the end of the solution to part a. should read as follows:

Each division on the x-axis corresponds to 5 doses or 5 days. Each division on the y-axis corresponds to 10 mg of medication.

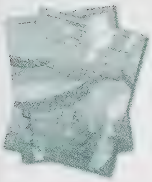
3. For each sequence, determine the common difference or common ratio; then state whether it is arithmetic or geometric.

   a. 3, 15, 27, 39, 51, …
   b. $-4, -0.8, -0.16, -0.032$

4. Turn to pages 270 to 273 of the textbook and answer the following.

   a. exercises 1, 2, and 3 of "Discussing the Ideas."
   b. exercises 1, 2, and 6 of "Exercises: Checking Your Skills"
   c. exercise 9 of "Exercises: Extending Your Thinking"

Compare your responses with the suggested answers in the Appendix, Activity 2, pages 58–66.



Turn to pages 4 to 7 of Assignment Booklet 6A and answer questions 5, 6, and 7.

Project: Building the Best Soft-Drink Container

ubiquitous: being or seeming to be everywhere at the same time

Example: aluminum soft drink cans

The soft-drink industry has worked very hard to build better and less-expensive containers. The all-aluminum can was introduced in 1964. There have been many improvements to its design over the years. For example, manufacturers have reduced the weight of their containers by more than a third over the past 30 years. Do you think you can design a container that serves consumers better?

Turn to page 114 of the Project Book and read "The Task" and "Background." If you have access to the Internet, the following website may give you some insight into the factors involved in designing a container:

http://list.gatech.edu/archives/lcc3020a1/0007.html

5. Answer exercises 1, 2, and 3 of "Getting Started" on pages 116 and 117 of the Project Book.

6. What human factors should influence the size and shape of a soft-drink container?

7. How does the shape of a container affect the cooling of the contents of the container in a refrigerator?

Compare your responses with the suggested answers in the Appendix, Activity 2, pages 67–71.

Turn to page 8 of Assignment Booklet 6A and answer question 8.

Looking Back

In this activity you modelled situations with sequences. Geometric sequences with a common ratio for successive elements are often useful in describing real-life phenomena. These sequences are related to exponential equations, where the common ratio is the base in the equation. If the first term of a geometric sequence is a and the common ratio is r, the equation related to the sequence is $y = ar^x$.

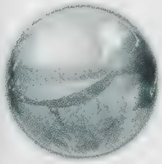
8. Turn to page 273 of the textbook and answer "Communicating the Ideas."

Compare your response with the suggested answer in the Appendix, Activity 2, page 71.

Turn to pages 9 to 12 of Assignment Booklet 6A and answer questions 9 to 14.

ACTIVITY 3

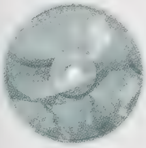
Introduction to Fractals



Nature provided the inspiration for Benoit Mandelbrot's study of fractals. Benoit Mandelbrot was largely responsible for the present interest in Fractal Geometry. He showed how Fractals can occur in many different places in both mathematics and in Nature. To find out more about Benoit Mandelbrot, visit the following Internet site:

http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Mandelbrot.html

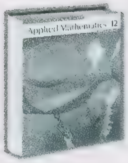
Fractals are objects that appear to be made up of smaller copies of themselves down to infinitesimal size. This may not seem to be wonderfully useful, but fractals are used in films and television to build realistic, seemingly unnatural landscapes and worlds. Sometimes, it's not clear how the final image is built up or even what the starting point was, but fractal images are often built up by applying a set of **iterative** or **recursive** rules. A starting image is modified by working at greater and greater detail until the final picture is completed. If you enlarge a part of the image, the details you will see are similar to that of the whole image.



To find out more about fractals, view the segment *Fractals* on the Applied Mathematics 30 CD.

In this activity you will create **self-similar** images by iteratively applying a given series of steps.

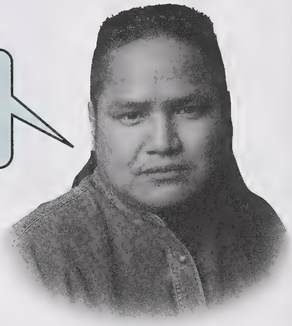
Turn to page 276 of the textbook and read the introductory paragraphs of Tutorial 6.3, "Introduction to Fractals."



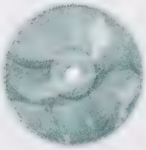
1. Complete exercises 1 to 9 of "Investigation 1: Create a 3-Dimensional Fractal" on pages 276 and 277 of the textbook.

Compare your responses with the suggested answers in the Appendix, Activity 3, pages 71–72.

Now, turn to page 278 of the textbook and read the paragraph at the top of the page.



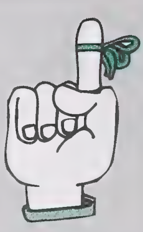
Computers are good at following instructions. This makes them ideal for generating images that can be defined using a set of rules.



From the Applied Mathematics 30 CD, you need to load the program *Logo* onto your computer. You will be using *Logo* to create simple and complex fractal designs. Click on the LOGO button at the bottom left of the menu.

For Macintosh users, follow the intstructions that appear on the screen.

For Windows users, double-click on the file *LoadLogo.html*. (It may simply appear as *LoadLogo* if you have viewing extensions turned off.) Follow the instructions given in this file to install *Logo* onto your computer.



Make sure you keep track of the folder that holds *ucblogo*. Copy the files *box.lg*, *hat.lg*, *koch.lg*, *sc.lg*, *sg.lg*, and *sr.lg* into the folder that contains *ucblogo*. (Again, they may show as *box*, *hat*, *koch*, *sc*, *sg*, and *sr* if you have viewing extensions turned off.) These files will provide additional capabilities to *ucblogo*.

Logo is a computer language developed to be simple to learn. It allows students from kindergarten to graduate school to use it for developing programs of considerable complexity. In this module, you will be using a very small part of the language. *Logo* uses a metaphor to describe its method of drawing. Drawings are carried out by a turtle dragging a pen. The programmer (you) tells the turtle which way to turn, how far to go, and when to draw with the pen and when not to. Your first task will be to create several iterations leading toward a Koch snowflake.

Start the *ucblogo* program. To exit this program, type the command `bye`. You can stop an active program by pressing the control key (CTRL) in combination with the Q key. (Macintosh users press the Command and Period keys at the same time.) When the program has started, you should have a question mark on the screen. This tells you that the turtle is waiting to do your bidding.

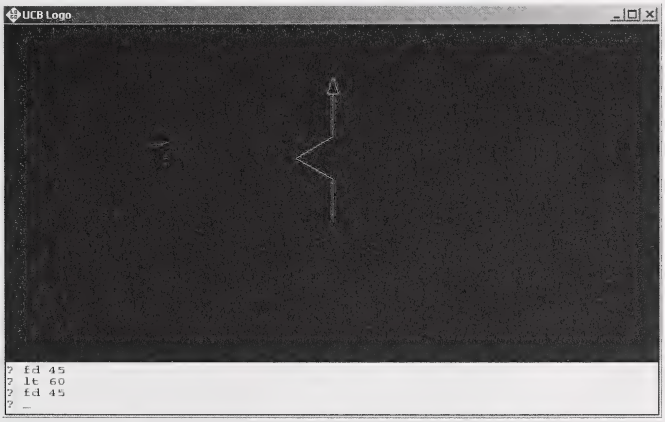
The Koch snowflake is based on an equilateral triangle. At each iteration (or step) in its development, a straight-line segment is replaced with a "bumpy line" consisting of four shorter segments.

Type the following commands, pressing ENTER after each one so you can observe what the turtle does. You'll see that the turtle in this version of *Logo* looks like an isosceles triangle. It points in the direction that the turtle is ready to walk.

```
? fd 45
? lt 60
? fd 45
? lt 240
? fd 45
? lt 60
? fd 45
```

Note: In the commands listed here, a space is required between the command letters and the number.

When you are finished with these commands, you should have a shape like the one shown here.



Did you notice that `fd` followed by a number makes the turtle move forward and that `lt` followed by a number changes the turtle's direction by rotating it counterclockwise? The instructions you entered earlier told the turtle to move forward 45 units, then turn left 60°, then move forward 45 units, then turn left 240°, then move forward 45 units, then turn left 60°, and then, finally, move forward 45 units.

The following table gives a number of commands that *Logo* understands and a short explanation of what they do.

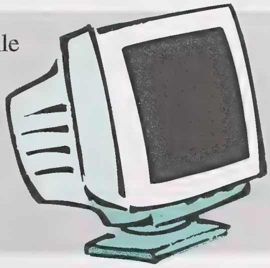
Command	Explanation
fd or forward	Move the turtle forward. This command must be followed by a number that tells the turtle how far forward to move.
bk or back	Move the turtle backward. This command must be followed by a number that tells the turtle how far backward to move.
lt or left	Change the direction the turtle is facing by rotating it counterclockwise. This command must be followed by a number that tells how many degrees the turtle should turn.
rt or right	Change the direction the turtle is facing by rotating it clockwise. This command must be followed by a number that tells how many degrees the turtle should turn.
pu or penup	Lift up the pen. This means that the turtle's path will not be traced.
pd or pendown	Put down the pen. This means that the turtle's path will be traced.
cs or clearscreen	Erase the screen to the background colour (usually black.)
ht or hideturtle	Hide the turtle. You cannot see the turtle, but it will still carry out your commands.
st or showturtle	Show the turtle.
setxy	Put the turtle at the *x*- and *y*-coordinates that must follow this command. The centre of the screen is $(0,0)$. For example, `setxy 10 20` will move the turtle to a position 10 pixels to the right of centre and 20 pixels above centre.
setpos	This is similar to setxy except the coordinates are placed inside brackets. e.g., `setpos [10 20]`
setheading	Tell the turtle what direction to face. A heading of 0 is straight upward, 90 is facing the right side of the screen, and so on.
heading	Give a number that tells which direction the turtle is heading.
make	Store a value in a variable. For example, `make "old 95` will store the value 95 in a variable called *old*.
stop	Stop a procedure.
fill	If the turtle is in an enclosed shape like a square or triangle, the shape will be filled with the current colour.

if	Allow decisions to be made based on some condition. e.g., `if (:old > 100) [make "really_old 1]`
and	This is a logical joining command that requires two conditions and is true only if both of the conditions are true. For example, `and (:b > 3) (:b < 10)` would be true only when b is between 3 and 10, not including 3 or 10.
not	This is a logical negation. Turns true to false and false to true. For example, `not (:b > :c)` would be true if b's value were not larger than c's value.
repeat	This allows a set of commands to be repeated. e.g., `repeat 3 [fd 45 rt 120]`
for	This allows a set of commands to be repeated and lets the number of repetitions that have occurred influence the commands. e.g., `for [i 1 6 1] [fd :i*10 rt 30]`
to	This is used to start the definition of a procedure.
end	This is used to conclude the definition of a procedure.

It gets to be a painstaking process to retype sets of commands over and over. *Logo* allows you to give a name to a set of commands and not have to type those commands again; you only need to use the name. This is known as defining a procedure. Type the following procedure into *Logo*. You will notice that the Bumpy Line procedure calls itself. This is known as recursion. *Logo* adds the commands on the lines after the initial `to` into the new procedure. *Logo* stops adding commands after the command `end`. (The list `[i 0 3 1]` in the following program ends with the digit 1, not the letter l.)

```
?to bumpy_line :s :c
>if (:c = 0) [fd :s stop]
>make "myheading heading
>for [i 0 3 1] [bumpy_line :s/3 :c-1 lt 60+:i*180]
>setheading :myheading
>end
```

Note: If you notice that you had made an error while entering the procedure, you will have to re-enter the entire procedure again. If you only notice that an entry error has been made because the procedure does not function properly, you will have to either restart *Logo* or use a new name for the procedure and re-enter the procedure.



Test the Bumpy Line procedure by entering `bumpy_line 81 3`. Your display should look like the following.

Now, type this second procedure into *Logo*. This procedure allows you to build several iterations of the Koch snowflake.

```
? to koch :c
> make "s 243
> pu
> setpos [0 150]
> setheading 150
> pd
> repeat 3 [bumpy_line :s :c rt 120]
> end
```

You will have to decide how many iterations to draw when you use this procedure. To use the koch procedure, type `koch` followed by a small natural number (the number of iterations to perform). You might want to try `koch 1` or `koch 3`. Numbers larger than 6 will likely take a very long time and will not display any additional detail. **Note:** If you try a number larger than 6 and it is taking along time, press CTRL Q to get you out.

On the Applied Mathematics 30 CD, there is a file called *koch.lg*, which contains versions of these two procedures. To use these procedures, type `bye` at the prompt (?) in *Logo*; then restart *Logo*. You can now type `koch 0` to draw the initial triangle, `koch 1` to draw the first iteration of the Koch snowflake, and so on. If you are interested in the commands used in making the snowflake, open *koch.lg* with any text editor, such as *Microsoft® NotePad*. This set of procedures uses commands from the chart given earlier and has numerous comments to help you understand how things are done.

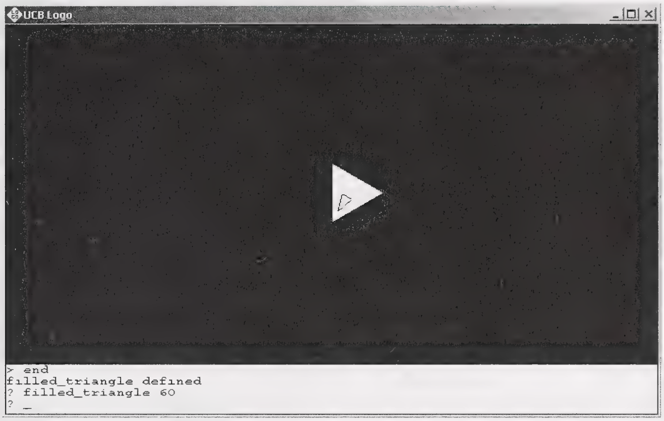
2. Use the koch procedure to create the first, second, and third iterations of the Koch snowflake. Sketch the results of the first and second iterations.

Compare your responses with the suggested answers in the Appendix, Activity 3, page 73.

Enter the following procedure into *Logo*. This will be the starting point for drawing a Sierpinski gasket (or Triangle).

```
? to filled_triangle :s
> repeat 3 [fd :s rt 120]
> lt 150
> pu
> bk :s/2
> fill
> end
```

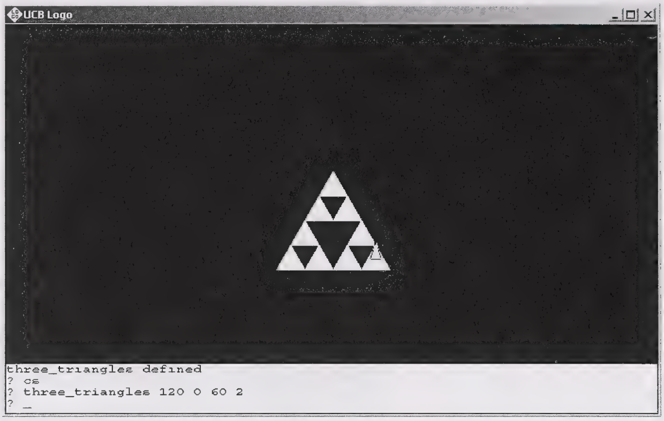
Test this procedure by typing `filled_triangle 60`. Your display should look like the following.

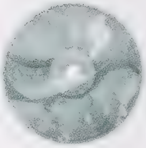


The Filled Triangle procedure is used in the following Three Triangles procedure. Enter this procedure into *Logo*.

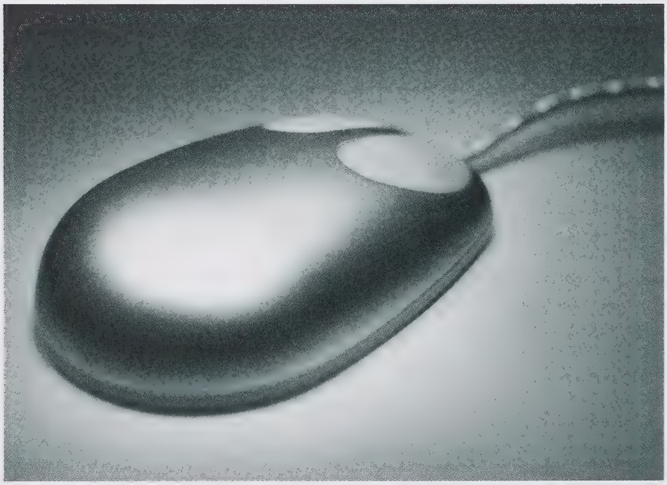
```
? to three_triangles :s :x :y :c
> pu
> setxy :x :y*sin(60)
> setheading 150
> pd
> if (:c = 0) [filled_triangle :s stop]
> three_triangles :s/2 :x  :y :c-1
> three_triangles :s/2 :x-:s/4 :y-:s/2 :c-1
> three_triangles :s/2 :x+:s/4 :y-:s/2 :c-1
> end
```

Test this procedure by typing `three_triangles 120 0 60 2`. Your display should look like the following.

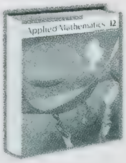




On the Applied Mathematics 30 CD, in the LOGO folder, there is a file called *sg.lg*. This file contains procedures to draw a Sierpinski gasket. You should have copied this file into the folder created where you installed *ucblogo* earlier. (If you did not, type `bye` beside the prompt (?), copy the file and restart *Logo*.) You can now type `sg 0` to draw the initial triangle, `sg 1` to draw the first iteration of the Sierpinski gasket, and so on. Again, if you are interested in the commands used in making the gasket, you can open *sg.lg* with any text editor, such as *Microsoft® NotePad*. This set of procedures uses commands from the chart given earlier and has numerous comments to help you understand what is going on.



3. Use the Sierpinski procedure (*sg*) to create the first, second, and third iterations of the Sierpinski gasket. Sketch the results of the first and second iterations.



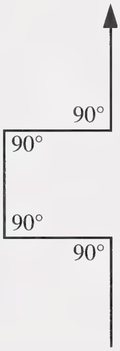
4. **a.** Turn to page 281 of the textbook and answer exercise 1 of "Discussing the ideas."

   b. What sequences can you discover in the Sierpinski gasket?

   c. What happens to the length of the Koch snowflake at greater recursive depths (after four or five iterations)?

   d. Why are self-similar patterns particularly suited to calculator or computer programs?

5. Modify the following Bumpy Line procedure to create a shape like the one on the right. Call your new procedure the Hat procedure. **Hint:** There are five segments, not four, and a 90° turn, not a 60° turn.



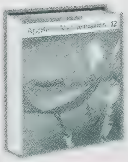

```
?to bumpy_line :s :c
>if (:c = 0) [fd :s stop]
>bumpy_line :s/3 :c-1 lt 60
>bumpy_line :s/3 :c-1 rt 120
>bumpy_line :s/3 :c-1 lt 60
>bumpy_line :s/3 :c-1
>end
```

6. Turn to page 282 of the textbook and answer exercise 3 of "Exercises: Checking Your Skills."

Compare your responses with the suggested answers in the Appendix, Activity 3, pages 74–80.

Looking Back

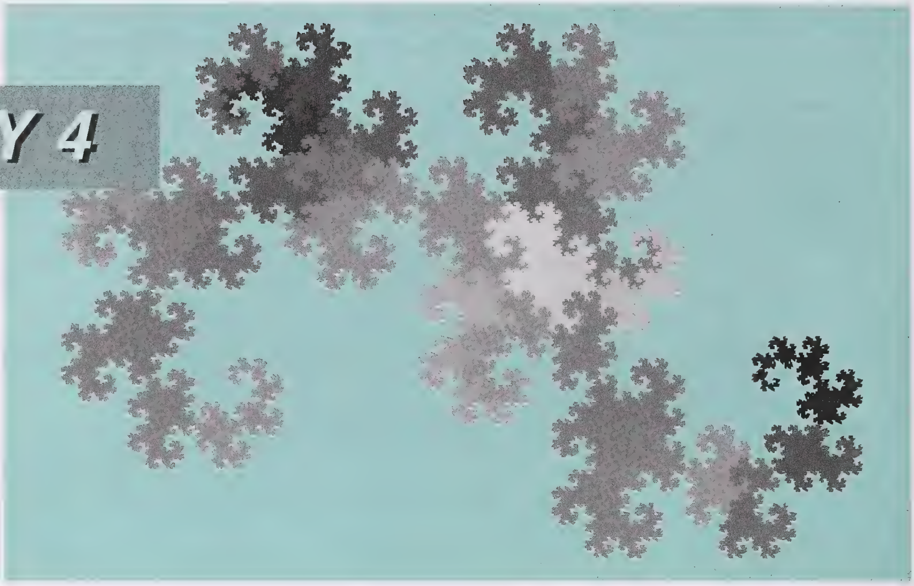
In this activity you studied fractals, recursion, and iterative processes. You witnessed how recursion can create images that are self-similar and how strangely beautiful these images can be.



7. Turn to page 282 of the textbook and answer "Communicating the Ideas."

Compare your response with the suggested answer in the Appendix, Activity 3, page 80.

ACTIVITY 4



Some Properties of Fractals

As well as being delicately beautiful, fractals have some almost unbelievable properties. They get their fractal name from one of the properties that is truly surprising. Have you heard the statement, “We live in a 3-dimensional world” (or perhaps a 4-dimensional world if time is included)? With advances in theoretical physics, you might have even heard that the 11-dimensional string theory will provide a single unified theory of matter. These whole-number dimensions have been used by mathematicians from ancient times to the present to describe the world. The objects they were dealing with were solid, everyday things, such as stones, trees, and land. Mathematicians in the late 1800s and early 1900s started looking at shapes that didn’t fit this solid, everyday pattern. They found shapes that seemed to be somewhere between 1- and 2-dimensional, such as the image at the start of this activity. This fractional dimension gave rise to the name fractal, although the use of the name fractal didn’t happen until much closer to the end of the twentieth century.

In this activity, you will study some of the unusual properties of fractals. You can start by turning to page 283 of the textbook and reading the introductory paragraphs of Tutorial 6.4, “Some Properties of Fractals.”

1. Complete the following exercises of “Investigation 1: Properties of a Koch Snowflake” on pages 284 and 285 of the textbook.

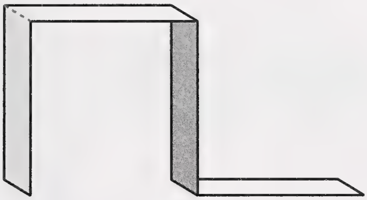
- **a.** exercises 2, 3, and 4 of “Part A: The Perimeter of a Koch Snowflake.”
- **b.** exercises 1 to 5 of “Part B: The Area of a Koch Snowflake.”

Compare your responses with the suggested answers in the Appendix, Activity 4, pages 81–86.

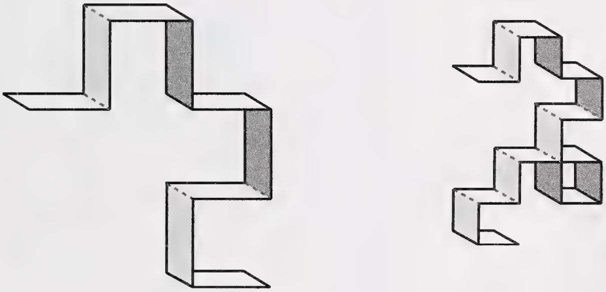
The Koch snowflake has an infinitely large perimeter even though it will fit inside a finite circle. You can get an idea of how things can be packed together tightly without filling up the space completely by doing the following paper folding exercise.

Step 1: Cut a 1-inch wide strip from the long side of a piece of notepaper. Fold the paper in half lengthwise to the right. Now, unfold it enough that the angle at the fold is only 90°. You should have a shape much like the one on the right.

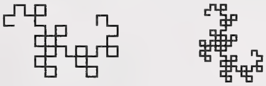
Step 2: Refold the piece of paper and fold it in half again, folding it the same way as in Step 1. Again, unfold it so that each fold is 90°



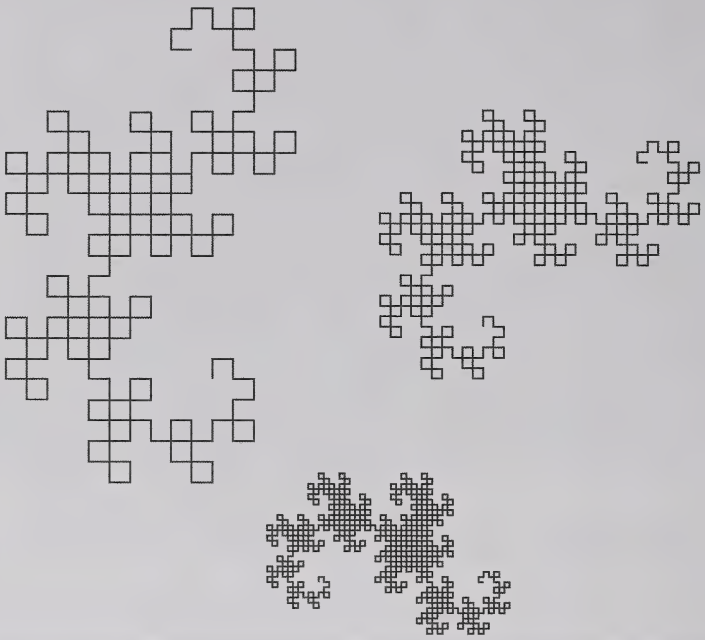
Step 3: Repeat this process another two times. You should get results like the following.



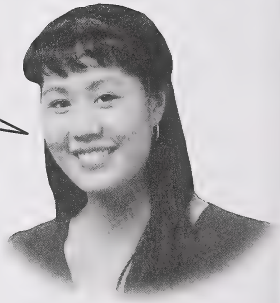
Notice how the amount of space needed to hold the folded paper decreases even though the length of the paper has not changed. You can try to fold the paper more times, but it is difficult to get more than six or seven folds made. If you were able to make more folds, the results would be like these images.

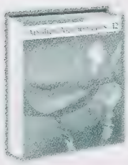


The following images, which appear after the preceding images, have been enlarged so the 90° angles can be seen.



Notice that the shape developing is the same as the one at the beginning of this activity.





2. Turn to pages 285 and 286 of the textbook and complete the following exercises of "Investigation 2: Some Properties of a Sierpinski Gasket."

   a. exercises 1, 2, and 3 of "Part A: The Perimeter of a Sierpinski Gasket"
   b. exercises 1 to 5 of "Part B: The Area of Sierpinski Gasket"

Compare your responses with the suggested answers in the Appendix, Activity 4, pages 86–89.

Turn to page 1 of Assignment Booklet 6B and answer questions 1 and 2.

You may wish to refer to the following website for information about fractional dimensions. It uses the Sierpinski Gasket as an illustrative example.

http://math.rice.edu/~lanius/fractals/dim.html

Turn to pages 287 and 288 of the textbook and carefully work through "Example: A Sequence of Cylinders."

3. Answer the following on pages 288 to 291 of the textbook.

a. exercises 1, 2, and 3 of "Discussing the Ideas"

b. exercises 1.a., 1.b., 2, 3.a., 3.b., and 4 of "Exercises: Checking Your Skills"
Note: The formula for volume of a sphere given in the glossary of the textbook is incorrect. It should read $V = \frac{4}{3}\pi r^3$.

c. exercise 6 of "Exercises: Extending Your Thinking"

Compare your responses with the suggested answers in the Appendix, Activity 4, pages 89–95.

Turn to pages 2 and 3 of Assignment Booklet 6B and answer questions 3 and 4.

Looking Back

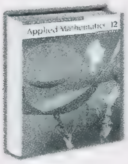
In this activity you studied some properties of fractals. You saw how the area of a fractal can continuously decrease while the perimeter increases (Sierpinski gasket). You also saw how both the area and perimeter can increase, where the area remains finite and the perimeter increases to infinity (Koch snowflake).

4. Turn to page 291 of the textbook and answer "Communicating the Ideas."

Compare your response with the suggested answer in the Appendix, Activity 4, page 96.

Turn to pages 3 to 6 of Assignment Booklet 6B and answer questions 5 and 6.

Module Review



This module dealt with Chapter 6: Patterns in the *Addison-Wesley Applied Mathematics 12 Source Book.*

Turn to page 294 of the textbook and review the skills and concepts that were developed in this module. Also, read the important results and formulas that you discovered.

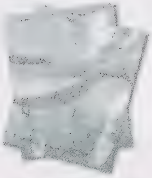
Answer the following exercises on pages 295 to 298 of the textbook.

1. exercises 1 and 2 of Part A of "What Should I Be Able to Do?"

Note: The heading of the last column in the table in exercise 2.a. should read "Total Perimeter (cm)," not "Increase in Perimeter (cm)."

2. exercises 4, 6, and 8 of Part B of "What Should I Be Able to Do?"

Compare your responses with the suggested answers in the Appendix, Module Review, pages 96–103.



Turn to pages 6 to 17 of Assignment Booklet 6B and complete the Module Review Assignment.

If you had difficulties understanding the skills and concepts in Module 6: Patterns, it is recommended that you contact your teacher for some extra help activities. If you have a clear understanding of the skills and concepts in this module, you may wish to do the following enrichment activity. You may decide to do both.

Enrichment

In the Appendix, the solutions for sums of geometric sequences use a trick. This trick can be used in a more general form to give a formula for the sum of a geometric sequence. If you have the geometric sequence $ar^0, ar^1, ar^2, ar^3, \ldots, ar^n$, the sum can be calculated just as it is when there is a numerical example. Let SUM be the sum of these $n+1$ terms of the sequence. Apply the same trick to a general geometric sequence, to get the two equations that follow, where *a* is the initial term and *r* is the common ratio.

$$\text{SUM} = ar^0 + ar^1 + ar^2 + ar^3 + \ldots + ar^n$$

$$\begin{aligned} r \times \text{SUM} &= r \times ar^0 + r \times ar^1 + r \times ar^2 + r \times ar^3 + \ldots + r \times ar^n \\ &= ar^1 + ar^2 + ar^3 + \ldots + ar^n + ar^{n+1} \end{aligned}$$

If the last equation is subtracted from the first equation, a new equation can be obtained.

$$\begin{array}{rl} \text{SUM} = ar^0 + ar^1 + ar^2 + ar^3 + \ldots + ar^n & \text{①} \\ r \times \text{SUM} = \quad ar^1 + ar^2 + ar^3 + \ldots + ar^n + ar^{n+1} & \text{②} \\ \hline \text{SUM} - r \times \text{SUM} = ar^0 + 0 + 0 + 0 + \ldots + 0 - ar^{n+1} & \text{①} - \text{②} \end{array}$$

This new equation can be written in a simpler form to give a formula for the SUM of the first $n+1$ terms of a geometric sequence.

$$\begin{aligned} \text{SUM} - r \times \text{SUM} &= ar^0 - ar^{n+1} \\ \text{SUM}(1-r) &= a\left(1-r^{n+1}\right) \\ \text{SUM} &= \frac{a\left(1-r^{n+1}\right)}{1-r} \end{aligned}$$

What are the sums of the following sequences or parts of sequences?

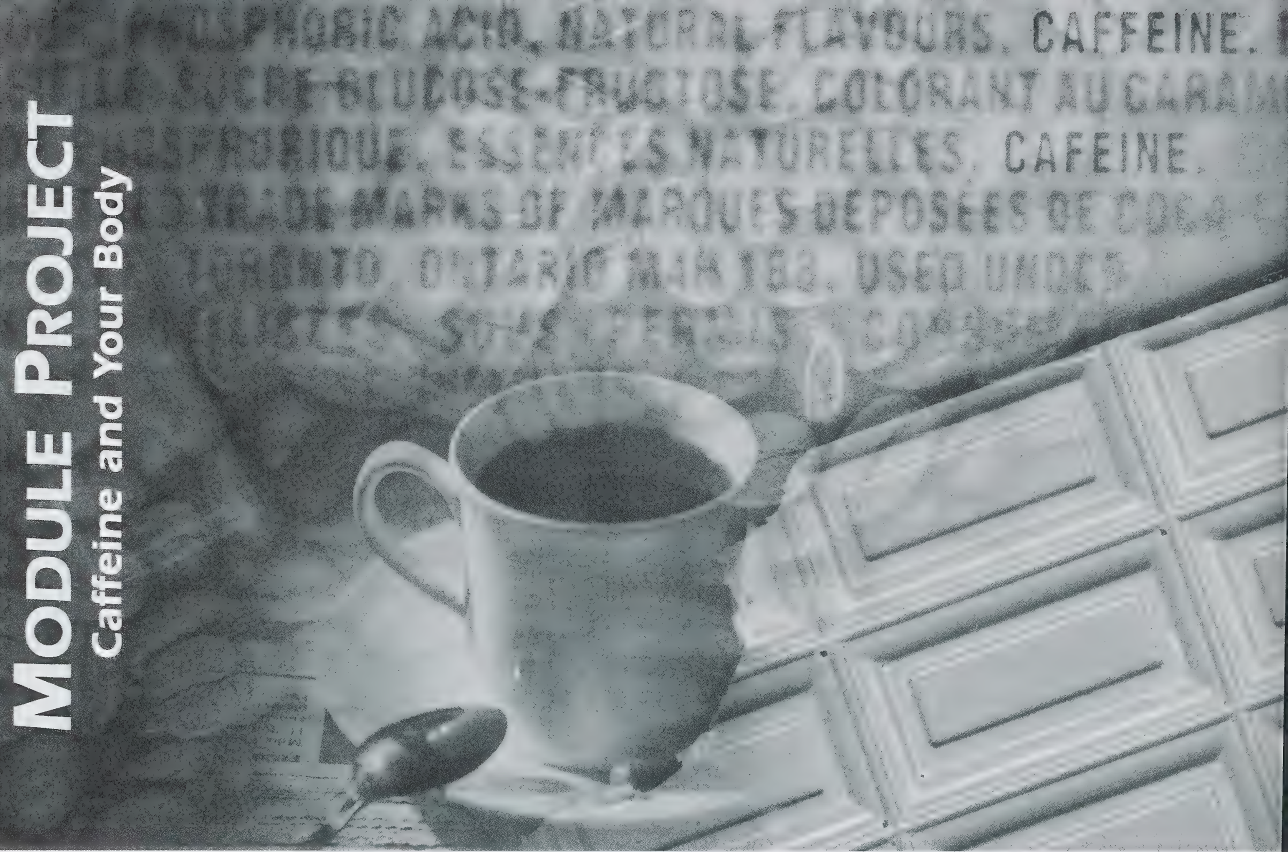
1. the first ten terms of the sequence 2, 4, 8, 16, 32, 64, 128, …

2. the sequence 8, 4, 2, 1, …

3. the first 25 terms of the sequence 4, 12, 36, …

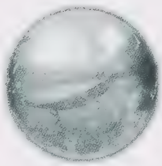
4. the sequence 0.7, 0.07, 0.007, …

Compare your responses with the suggested answers in the Appendix, Module Review: Enrichment, pages 104–105.



Completing the Project

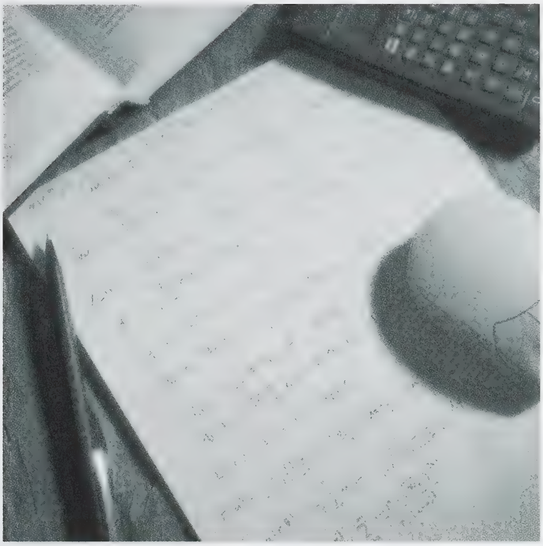
By now you should have completed the initial research for the module project, Caffeine and Your Body. You should have learned some of the sources of caffeine, how this drug affects your body, and what some of the side effects of using caffeine are. The following websites may give you some additional information.



- **http://www.howstuffworks.com/caffeine.htm**
- **http://www.cs.unb.ca/~alopez-o/Coffee/caffaq.html**
- **http://faculty.washington.edu/chudler/caff.html**
- **http://www.nlm.nih.gov/medlineplus/druginfo/caffeinesystemic202105.html**

Armed with the knowledge you have uncovered about caffeine, turn to pages 274 and 275 of the textbook and answer exercises 1 and 2 of "How Long Does Caffeine Remain in Your Body?" Complete the work on these pages by completing a report as described after exercise 2. A copy of this report will be required in the module project section of Assignment Booklet 6B. You should include a list of websites and other materials you used to obtain information. The report should be at least one page of textual material and another page showing graphs and other illustrations. You probably should intersperse the graphs and illustrations with your writing rather than have them on separate pages. Store your results and your report in the project section of your mathematics binder. More information about what is expected in the report is given in Assignment Booklet 6B.

Now, proceed to page 292 of the textbook and create a poster as described in "How Does Caffeine Affect Your Body?" Again, store your poster in the project section of your mathematics binder. Your poster should be at least 11 inches by 17 inches in size. (You can tape two 8.5-inch by 11-inch sheets of paper to create a large enough paper. This will make it easy to fold to fit with the Assignment Booklet 6B.)



Turn to pages 299 and 300 of the textbook and answer exercises 9 and 10 of Part C of "What Should I Be Able to Do?"

Compare your responses with the suggested answers in the Appendix, Module Project, pages 106–107.

Module Project

Now that you have worked through some exercises to help you gain insight into the module project, you may wish to revise information you have developed and stored in the project section of your mathematics binder. When you have finished your revisions, complete the module project, Caffeine and Your Body.

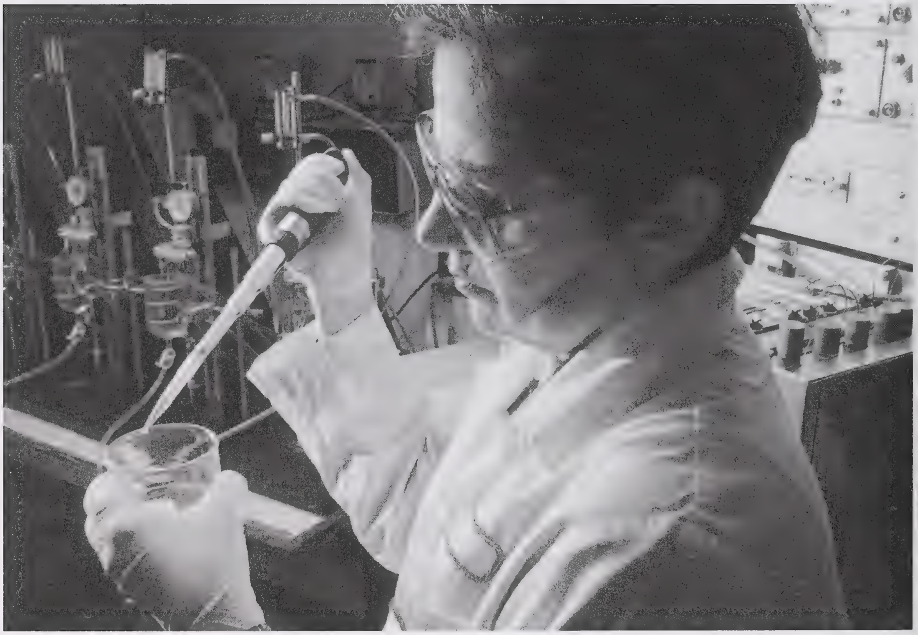
You should use your responses from the textbook exercises on pages 258, 274, 275, 292, 299, and 300 to assist you in the completion of the module project.

Turn to pages 18 to 22 of Assignment Booklet 6B and complete the module project.

Module Summary

In this module you examined patterns, fractals, and sequences. You identified arithmetic, geometric, and recursive sequences; used technology and recursive methods to generate sequences; and modelled problems using sequences. You then studied self-similarity and used it and iteration to create fractals. You were then shown how fractals can have strange properties, like infinite perimeter with a finite area, as you calculated the perimeter, area, and volume of fractals.

Understanding patterns and sequences can help you understand how drugs are metabolized and what length of time is required for a drug to reach its maintenance level. If you are interested in a career in the pharmaceutical industry, you will find that you will need to be able to perform calculations and analyses similar to those you performed in this module.



APPENDIX

Glossary
Suggested Answers
Image Credits

Glossary

arithmetic sequence: a sequence in which each successive term is found by adding a constant to the preceding term

bounded: the value of an expression is bounded if there is one pair of integers with the first always less than the expression and the second always larger than the expression

closed form: an expression with all terms showing

common difference: the difference between two consecutive terms of an arithmetic progression

common ratio: the ratio of each term of a geometric progression to the term preceding it

finite: bounded

fractal: an object that appears to be made up of smaller copies of the original

geometric sequence: a sequence in which the ratio of a term to its predecessor is the same throughout the sequence

infinite: not bounded

iteration: a repetition of a sequence of operations that yields results successively closer to a desired result

recursive: a procedure that can repeat itself indefinitely

recursively generated sequence: a sequence generated by a recursive process

sequence: a set of elements ordered so they can be labelled with the positive integers

without limit: not bounded or continuing forever

Suggested Answers

Activity 1: Sequences

1. The textbook explains that 1 potato provides 6 new plants (or eyes). Since 1 plant will give between 5 and 10 new potatoes after a single growing season, there will be enough potato eyes to start between 30 and 60 new plants for the next growing season. After two seasons, there will be between 150 and 600 new potatoes.

   The following table shows how the numbers increase over the first five growing seasons.

Season	New Plants Started	Minimum Number of Potatoes	Maximum Number of Potatoes	Minimum Number of "eyes"	Maximum Number of "eyes"
1	1	5	10	30	60
2	30 to 60	150 (5×30)	600 (10×60)	900 (6×150)	3600 (6×600)
3	900 to 3600	4500 (5×900)	36 000 (10×3600)	27 000 (6×4500)	216 000 $(6 \times 36\,000)$
4	27 000 to 216 000	135 000 $(5 \times 27\,000)$	2 160 000 $(10 \times 216\,000)$	810 000 $(6 \times 135\,000)$	12 960 000 $(6 \times 2\,160\,000)$
5	810 000 to 12 960 000	4 050 000 $(5 \times 810\,000)$	129 600 000 $(10 \times 12\,960\,000)$		

   After five growing seasons, one potato plant (or eye) could have anywhere between 4 050 000 and 129 600 000 potatoes as its descendants.

2. **Textbook exercises 1 to 8 of "Investigation 1: The Growth of Bacteria," p. 261**

   1. The first ten terms of the sequence describing the number of bacteria are 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512.

   2. The number of bacteria doubles every 30 minutes, which gives a geometric series. The ratio of successive pairs of the sequence is 2; there is a common ratio of 2.

Activity 1 (continued)

3.

	A	B	C
1	**Number of Generations**	**Number of Bacteria**	**Elapsed Time (h)**
2	1	1	0.00
3	2	2	0.50
4	3	4	1.00
5	4	8	1.50
6	5	16	2.00
7	6	32	2.50
8	7	64	3.00
9	8	128	3.50
10	9	256	4.00
11	10	512	4.50

	A	B	C
1	**Number of Generations**	**Number of Bacteria**	**Elapsed Time (h)**
2	1	1	0.0
3	=A2+1	=B2*2	=C2+0.5
4	=A3+1	=B3*2	=C3+0.5
5	=A4+1	=B4*2	=C4+0.5
6	=A5+1	=B5*2	=C5+0.5
7	=A6+1	=B6*2	=C6+0.5
8	=A7+1	=B7*2	=C7+0.5
9	=A8+1	=B8*2	=C8+0.5
10	=A9+1	=B9*2	=C9+0.5
11	=A10+1	=B10*2	=C10+0.5

4. Answers will vary depending on where you go to school and where you live. In 2001, the population of Alberta was slightly more than 3 million. Therefore, it would take 23 generations or 11 h for the bacteria to at least equal the population of Alberta.

5. In 2001, Canada's population was just over 31 million. Therefore, it would take 12.5 h for the number of bacteria to exceed this number.

6.

	A	B	C	D
1	Number of Generations	Number of Bacteria	Elapsed Time (h)	Mass (kg)
2	1	1	0.00	0.000001
3	2	2	0.50	0.000002
4	3	4	1.00	0.000004
5	4	8	1.50	0.000008
6	5	16	2.00	0.000016
7	6	32	2.50	0.000032
8	7	64	3.00	0.000064
9	8	128	3.50	0.000128
10	9	256	4.00	0.000256
11	10	512	4.50	0.000512
12	11	1,024	5.00	0.001024
13	12	2,048	5.50	0.002048
14	13	4,096	6.00	0.004096
15	14	8,192	6.50	0.008192
16	15	16,384	7.00	0.016384
17	16	32,768	7.50	0.032768
18	17	65,536	8.00	0.065536
19	18	131,072	8.50	0.131072
20	19	262,144	9.00	0.262144
21	20	524,288	9.50	0.524288
22	21	1,048,576	10.00	1.048576
23	22	2,097,152	10.50	2.097152
24	23	4,194,304	11.00	4.194304
25	24	8,388,608	11.50	8.388608
26	25	16,777,216	12.00	16.777216
27	26	33,554,432	12.50	33.554432
28	27	67,108,864	13.00	67.108864
29	28	134,217,728	13.50	134.217728

7. Answers will vary. For an 80-kg student, the total mass of the bacteria will be equal some time between 13 h and 13.5 h.

8. The world is not consumed by bacteria because they not only multiply, they die. Not only do other organisms and toxins kill the bacteria, bacteria also require the necessary nutrients to stay alive to continue doubling every 30 minutes.

3. **Textbook exercises 7, 8, and 9 of "Investigation 2: A Bouncing Ball," pp. 262 and 263**

7. and 8. Answers will vary. The sample answers given are based on the results provided with the question. **Note:** If you were able to use the *Ball Bounce 1* explorer and used the information from the initial trial provided, the value listed under 1 is the height of the first bounce when the program first begins. For subsequent uses, the value listed under 1 is the height from which the ball was dropped. The following table uses the results given in the exercise.

Bounce	Maximum Height	Difference in Heights	Ratio of Heights
1	19.6 m		
2	14.4 m	−5.2 m	0.73
3	10.0 m	−4.4 m	0.69
4	7.2 m	−2.8 m	0.72
5	4.9 m	−2.3 m	0.68

9. **a.** There doesn't appear to be a common difference in successive heights.

b. The ratios of successive heights are not all the same, but they seem to group around 0.7. In experiments, measurement errors are expected, thus making the ratios vary slightly.

c. The height is better described by a geometric sequence. Therefore, height of bounce 6 would be $4.9 \times 0.7 \doteq 3.4$ m. The initial data gives a value of 3.6 m, which is close to the prediction.

4. **Textbook exercises 2, 3, and 4 of "Discussing the Ideas," pp. 263 and 264**

2. This sequence could have been generated from the function $y = 2^{x-1}$, where x is the generation number.

3. One of five scenarios can occur if the common ratio is less than 1.

Scenario 1: If the common ratio is 0, the sequence simply sits at 0 after a non-zero first element.

Example: 7, 0, 0, 0, 0, …

Generally, this sequence is excluded because you cannot compute a ratio since dividing by zero is undefined.

Scenario 2: If the common ratio is between 0 and 1, each succeeding term decreases and approaches zero.

Example: $10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \frac{5}{8}, \ldots$

Scenario 3: If the common ratio is between -1 and 0, the terms of the sequence approach zero but oscillate from positive to negative.

Example: $10, -5, 2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8}, \ldots$

Scenario 4: If the common ratio is -1 the terms oscillate from positive to negative.

Example: $-3, 3, -3, 3, \ldots$

Scenario 5: If the common ratio is smaller than -1 the terms move farther from zero and oscillate from positive to negative.

Example: $1, -2, 4, -8, 16, -32, \ldots$

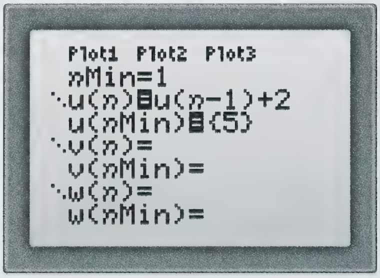
4. Exponential growth is likely to generate data that forms a geometric sequence. The defining relation for a geometric sequence is exponential in nature. This is shown in the following chart.

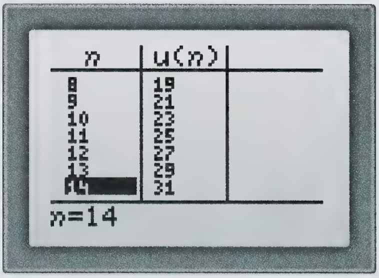
Term	Value	Common Ratio	Exponential Form
1	6		
2	12	2	6×2^1
3	24	2	6×2^2
4	48	2	6×2^3
5	96	2	6×2^4
6	192	2	6×2^5

You can see that whatever the common ratio, the value can be expressed in the exponential form.

5. a. Textbook exercises 1.a., 1.c., 1.e., 3, 4, and 7 of "Exercises: Checking Your Skills," pp. 264 and 265

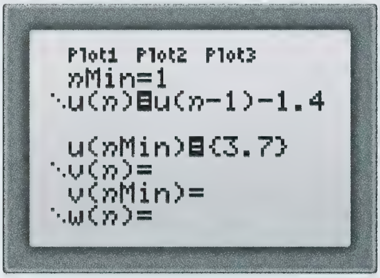
1. a.

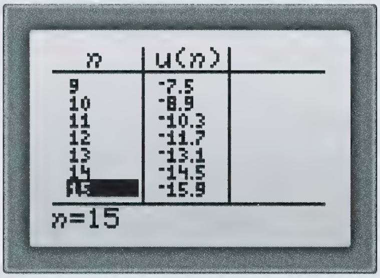




The 14th term is 31.

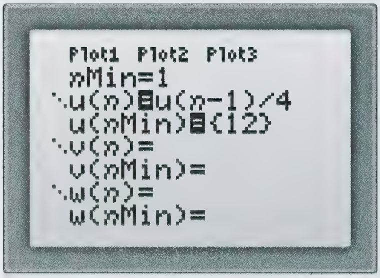
c.

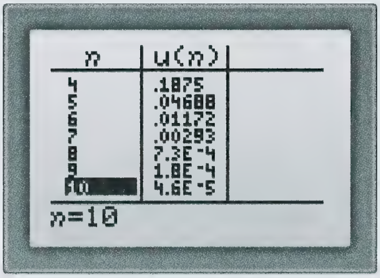




The 15th term is -15.9.

e.





The 10th term is 4.6×10^{-5}, or 0.000 046.

3. **a.**

Date	Insect Count
May 1	100
June 1	200
July 1	400

There will be 400 insects at the beginning of July.

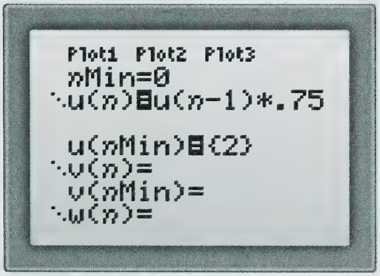
b.

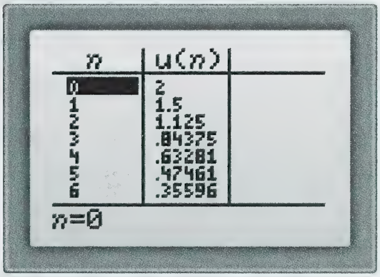
Date	Insect Count
May 1	100
June 1	200
July 1	400
August 1	800
September 1	1600

There will be 1600 insects at the beginning of September.

4. **a.** Answers may vary. Two sample answers are given.

Method 1: Using a Graphing Calculator





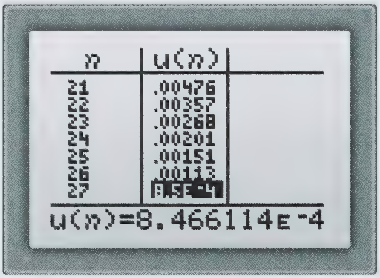
Method 2: Using a Spreadsheet

	A	B	C
1	**Bounce Number**	**Height (m)**	**Height (cm)**
2	0	2.00000	200.000
3	1	1.50000	150.000
4	2	1.12500	112.500
5	3	0.84375	84.375
6	4	0.63281	63.281
7	5	0.47461	47.461
8	6	0.35596	35.596
9	7	0.26697	26.697
10	8	0.20023	20.023
11	9	0.15017	15.017
12	10	0.11263	11.263

b. After each of the first five bounces, the ball reaches a height of 1.5 m, 1.125 m, 0.843 75 m, 0.632 81 m, and 0.474 61 m respectively.

c. The ball reaches a height of approximately 20 cm after 8 bounces.

d.

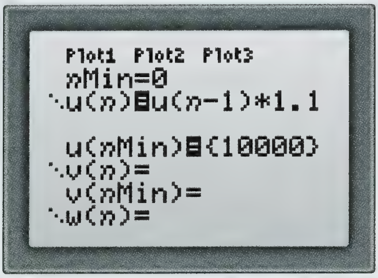


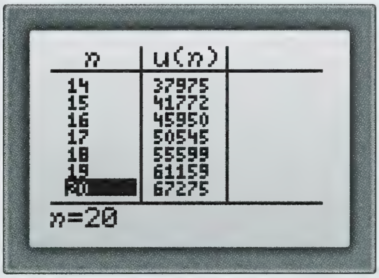
The 27th term shows that the ball reaches a height of less than 0.1 cm. Therefore, it will have bounced 26 times before this height is reached and it may be considered to have stopped after the 26th bounce. **Note:** If you are using a calculator, you will need to convert 8.5E$^-$4 m to centimetres.

7. a. The interest earned in the first year was $1000, which is 10% of $10 000. The interest earned in the second year was $1100, which is 10% of $11 000. Therefore, the annual interest rate was 10%.

b. The amounts form a geometric sequence.

c.





Jamie will have $67 275 in the account after 20 years.

b. Textbook exercise 10 of "Exercises: Extending Your Thinking," p. 265

10. The solution to this question is simpler to determine if these two questions are answered:

- "How far does the ball fall?"
- "How far does the ball rise?"

The sum of these distances is the total distance travelled.

Trips to Ground	Distance Risen	Distance Fallen	Total Distance
1	0	20 m	20 m
2	16 m	16 m	52 m
3	12.8 m	12.8 m	77.6 m
4	10.24 m	10.24 m	98.08 m
5	8.192 m	8.192 m	114.464 m

The ball has travelled 114.464 m when it hits the ground for the fifth time.

6. Textbook exercise "Communicating the Ideas," p. 265

Answers will vary. A sample answer is given.

In an arithmetic sequence, each pair of successive terms has the same difference.

Examples: 1, 3, 5, 7, 9, … (Each succeeding term increases by 2.)

$9, 6, 3, 0, -3, -6, \ldots$ (Each succeeding term decreases by 3.)

Activity 1 (continued)

In a geometric sequence, each pair of successive terms has the same ratio.

Examples: 1, 3, 9, 27, 81, … (Each succeeding term is 3 times the preceding term.)

$10, 5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$ (Each succeeding term is one-half the preceding term.)

A sequence that is neither arithmetic nor geometric has some other rule used to determine its elements.

Examples: 3, 1, 4, 1, 5, 9, … (Each succeeding element is the next digit of π.)

1, 2, 2, 3, 3, 3, 4, … (There are as many copies of a number as the magnitude of the number.)

$1, 1, 2, \frac{1}{3}, 4, \frac{1}{9}, 8, \frac{1}{27}, 16, \frac{1}{81}, \ldots$ (Two geometric sequences are interweaved.)

Activity 2: Modelling Using Sequences

1. Textbook exercises 1 to 5 of "Investigation 1: Graph a Sequence Using a Spreadsheet Program," pp. 266 and 267

1. Spreadsheets may vary slightly. A sample spreadsheet is given.

The column for the number of trees cut can only contain whole numbers. Because cutting 25% of the trees may not give a whole number, you must decide how you are going to handle this situation. There are three possible ways to handle a calculated value that isn't a whole number:

- Round the value to the nearest whole number. Over time, using this method will result in the correct number of trees cut.
- Round the value down to the nearest whole number. Over time, using this method will result in fewer trees cut.
- Round the value up to the nearest whole number. Over time, using this method will result in more trees cut.

Note: There are assumptions in this answer that the planted trees could be harvested after about 4 or 5 years. In reality, the trees planted would not be ready for harvesting for about 50 years from the time they were planted.

The following spreadsheet rounds the number of trees cut to the nearest whole number.

	A	B	C	D	E
1	**Harvesting Trees**				
2					
3		Harvesting rate	25%		
4					
5	**Year**	**Number at Beginning of Year**	**Number Cut**	**Number Planted**	**Number at End of Year**
6	1	7000	1750	1500	6750
7	2	6750	1688	1500	6562
8	3	6562	1641	1500	6421
9	4	6421	1605	1500	6316
10	5	6316	1579	1500	6237
16	11	6056	1514	1500	6042
17	12	6042	1511	1500	6031
18	13	6031	1508	1500	6023
19	14	6023	1506	1500	6017
20	15	6017	1504	1500	6013
21	16	6013	1503	1500	6010
33	28	6001	1500	1500	6001
34	29	6001	1500	1500	6001
35	30	6001	1500	1500	6001
36	31	6001	1500	1500	6001
37	32	6001	1500	1500	6001

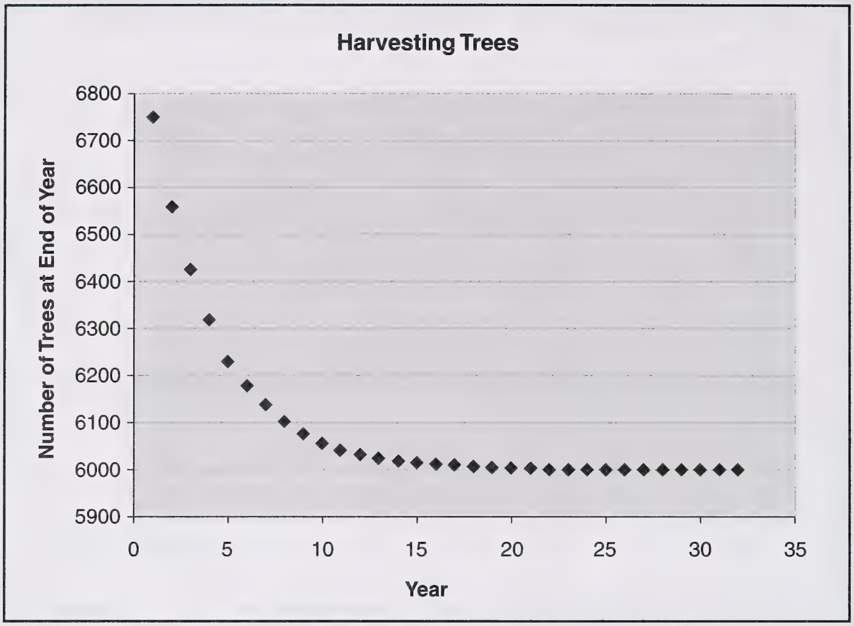
Here are the formulas used to create this spreadsheet.

	A	B	C	D	E
1	**Harvesting Trees**				
2					
3		Harvesting rate	0.25		
4					
5	**Year**	**Number at Beginning of Year**	**Number Cut**	**Number Planted**	**Number at End of Year**
6	1	7000	=ROUND(B6*$C$3,0)	1500	=B6-C6+D6
7	=A6+1	=E6	=ROUND(B7*$C$3,0)	=$D$6	=B7-C7+D7
8	=A7+1	=E7	=ROUND(B8*$C$3,0)	=$D$6	=B8-C8+D8

If you chose to round the values down, the formula in cell C6 should read =ROUNDDOWN(B6*$C$3,0) or =FLOOR(B6*$C$3,1). If you chose to round the values up, the formula in cell C6 should read =ROUNDUP(B6*$C$3,0) or =CEILING(B6*$C$3,1).

2. At the end of the thirteenth year, there are 6023 trees in the forest.

3.



4. Over time, the forest decreases to about 6000 trees and then stabilizes. This occurs at 22 years.

5.

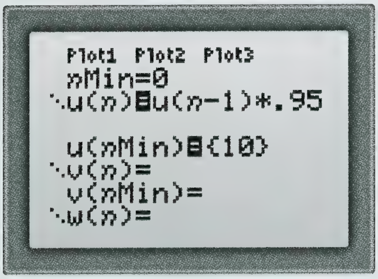
	A	B	C	D	E
1	**Harvesting Trees**				
2					
3		Harvesting rate	30%		
4					
5	**Year**	**Number at Beginning of Year**	**Number Cut**	**Number Planted**	**Number at End of Year**
6	1	7000	2100	1500	6400
7	2	6400	1920	1500	5980
8	3	5980	1794	1500	5686
9	4	5686	1706	1500	5480
10	5	5480	1644	1500	5336
22	17	5006	1502	1500	5004
23	18	5004	1501	1500	5003
24	19	5003	1501	1500	5002
25	20	5002	1501	1500	5001
26	21	5001	1500	1500	5001
27	22	5001	1500	1500	5001

To reach a stable forest population of 5000 over a period of 20 years, the trees can be harvested at a rate of 30% if the number of trees planted per year remains at 1500.

2. **Textbook exercises 1 to 10 of “Investigation 2: The Metabolism of Insulin,” pp. 267 and 268**

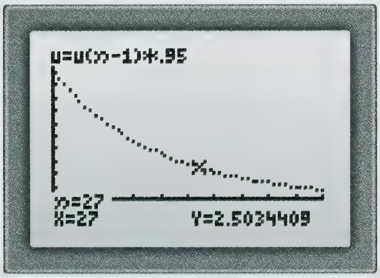
1. If 95% remains after one minute, a recursive rule for the amount of insulin left in the body is $u(n) = u(n-1) \times 0.95$, where n is the number of minutes since the injection.

2.



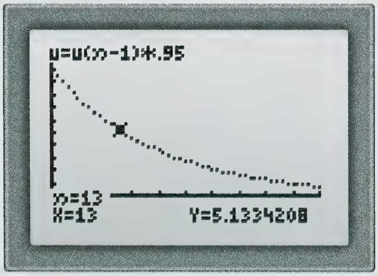
3. and 4. The values on the horizontal axis represent time (in minutes). The values on the vertical axis represent the number of units of insulin left in the body.

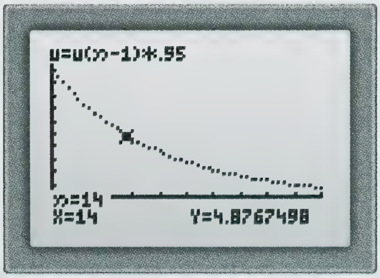
5. Press 2nd [CALC] 1 to choose the Value feature; then enter 27 for n and press ENTER.



After 27 minutes, about 2.5 units of insulin are left in the body.

6. Use the Trace feature to find the value of n that results in an insulin level of 5 units.



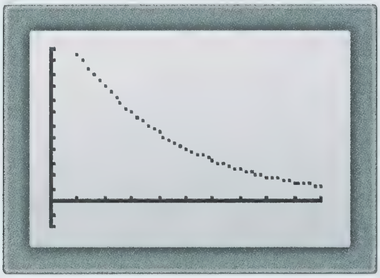


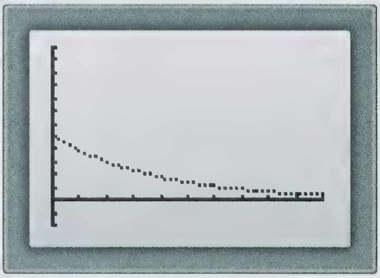
The half-life of insulin is between 13 minutes and 14 minutes. **Note:** Using the information from question 5, the quarter life is about 27 minutes. Therefore, to be more exact, the half-life of insulin would be about 13.5 minutes (half of 27 minutes).

7. $13 \times 5 = 65$ and $14 \times 5 = 70$

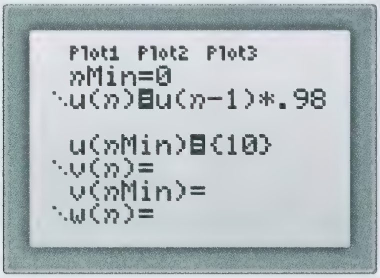
The drug will have effectively disappeared from the body between 65 to 70 minutes. Using the estimate of 13.5 minutes for half-life would give a time of 67.5 minutes for the insulin to have effectively left the body.

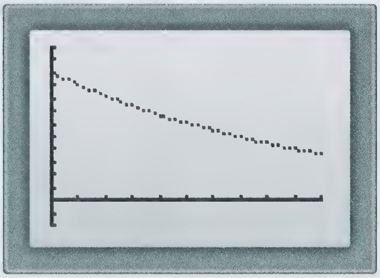
8. **Initial Dose of 15 Units** **Initial Dose of 5 Units**





9. The graph of the new drug that is eliminated more slowly looks like the following.





10. Insulin helps the body control the level of sugar in the blood. Diabetics often have to eat after an injection to raise their blood sugar because the injection causes the blood sugar to fall. The body doesn't function well if the blood sugar is outside a fairly limited range of values.

3. a.

Term	Difference	Ratio
3		
15	12	5
27	12	1.8
39	12	1.444
51	12	1.308

The common difference is 12. There is no common ratio. Therefore, this sequence is arithmetic.

b.

Term	Difference	Ratio
−4		
−0.8	3.2	0.2
−0.16	0.64	0.2
−0.032	0.128	0.2

There is no common difference, but there is a common ratio of 0.2. Therefore, this sequence is geometric.

4. a. **Textbook exercises 1, 2, and 3 of "Discussing the Ideas," p. 270**

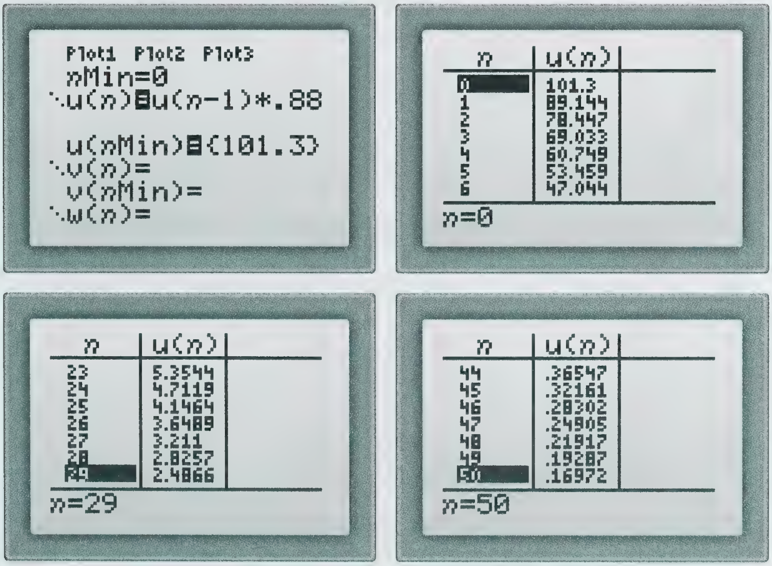
   1. A sequence is an ordered list of numbers. A function is a rule for turning one number into another.

   2. Geometric sequences are often used to model biological activity because they match the actual events closely.

   3. Using a spreadsheet allows you to quickly change values and build graphs that can be seen at the same time as the values.

b. **Textbook exercises 1, 2, and 6 of "Exercises: Checking Your Skills," pp. 270 and 271**

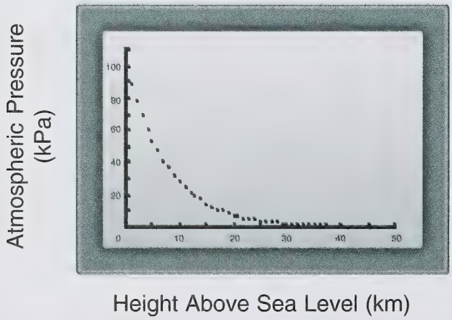
   1. a. **Method 1: Using a Graphing Calculator**



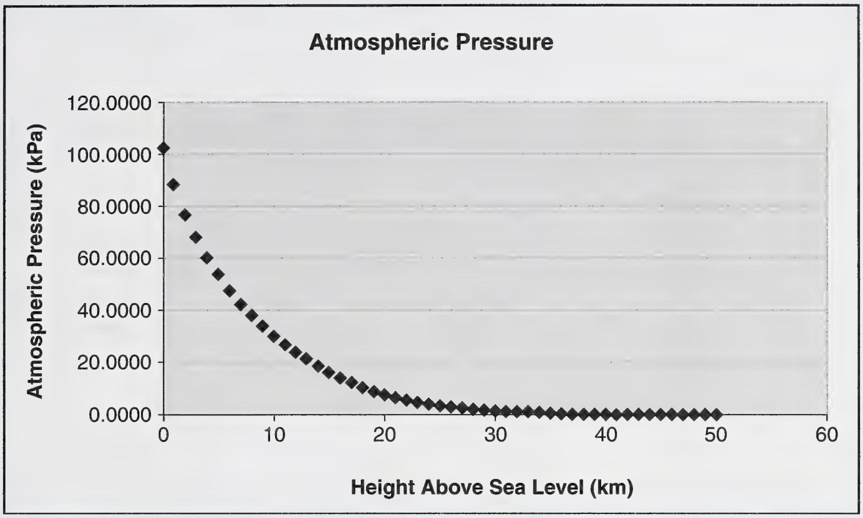
Method 2: Using a Spreadsheet

	A	B
1	Height Above Sea Level (km)	Atmospheric Pressure (kPa)
2	0	101.3000
3	1	89.1440
4	2	78.4467
5	3	69.0331
6	4	60.7491
7	5	53.4592
8	6	47.0441
9	7	41.3988
10	8	36.4310
11	9	32.0593
12	10	28.2121
42	40	0.6094
43	41	0.5363
44	42	0.4719
45	43	0.4153
46	44	0.3655
47	45	0.3216
48	46	0.2830
49	47	0.2491
50	48	0.2192
51	49	0.1929
52	50	0.1697

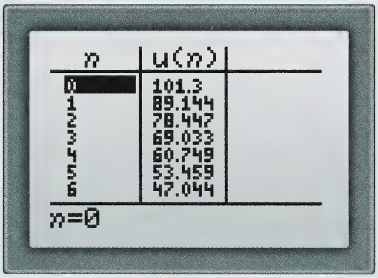
b. Method 1: Using a Graphing Calculator



Method 2: Using a Spreadsheet

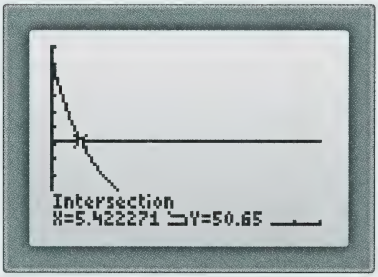


c.



Pressure will be half of that at sea level between 5 km and 6 km above ground.

This situation can also be modelled by the equation $y = 101.3 \times (0.88)^x$. Finding the intersection of this equation with the equation $y = \frac{101.3}{2}$ will give the point where the atmospheric pressure is exactly half that of the pressure at sea level.



According to the calculator, half the atmospheric pressure at sea level occurs at a height of approximately 5.4 km.

2. a. Because you cannot sell a partial seedling, make sure you program the spreadsheet to insert whole values. In this spreadsheet, the formula in cell C6 is =ROUND(86*$C$3,0).

	A	B	C	D	E
1	**Cherry Tree Seedlings**				
2					
3		Rate of sales	15%		
4					
5	**Year**	**Seedlings at Start of Year**	**Seedlings Sold**	**Seedlings Planted**	**Seedlings at End of Year**
6	1	6000	900	800	5900
7	2	5900	885	800	5815
8	3	5815	872	800	5743
9	4	5743	861	800	5682
10	5	5682	852	800	5630
11	6	5630	845	800	5585
12	7	5585	838	800	5547
13	8	5547	832	800	5515
14	9	5515	827	800	5488
15	10	5488	823	800	5465
16	11	5465	820	800	5445

b. There will be 5445 seedlings at the end of 11 years.

c.

	A	B	C	D	E
1	**Cherry Tree Seedlings**				
2					
3		Rate of sales	20%		
4					
5	**Year**	**Seedlings at Start of Year**	**Seedlings Sold**	**Seedlings Planted**	**Seedlings at End of Year**
6	1	6000	1200	800	5600
7	2	5600	1120	800	5280
8	3	5280	1056	800	5024
9	4	5024	1005	800	4819
10	5	4819	964	800	4655
21	16	4070	814	800	4056
22	17	4056	811	800	4045
23	18	4045	809	800	4036
24	19	4036	807	800	4029
25	20	4029	806	800	4023

At the end of 20 years, there will be 4023 seedlings remaining.

6. a.

	A	B	C
1	Year	Amount of Pollutant in Lake A (t)	Amount of Pollutant in Lake B (t)
2	0	250.00	75.00
3	1	272.50	97.50
4	2	292.98	122.03
5	3	311.61	148.39
6	4	328.56	176.44
7	5	343.99	206.01
18	16	444.72	600.28
19	17	449.69	640.31
20	18	454.22	680.78
21	19	458.34	721.66
22	20	462.09	762.91

B3=B2-0.09*B2+45
C3=C2+0.09*B2

Lake A will have about 343.99 t of pollutant after 5 years and about 462.09 t after 20 years. Lake B will have about 206.01 t of pollutant after 5 years and about 762.91 t after 20 years.

b.

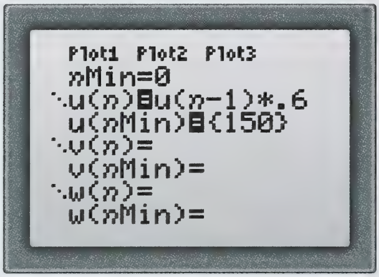
	A	B	C
1	Year	Amount of Pollutant in Lake A (t)	Amount of Pollutant in Lake B (t)
2	0	250.00	75.00
3	1	272.50	97.50
4	2	292.98	122.03
114	112	499.99	4865.01
115	113	499.99	4910.01
116	114	499.99	4955.01
117	115	500.00	5000.00
118	116	500.00	5045.00
119	117	500.00	5090.00

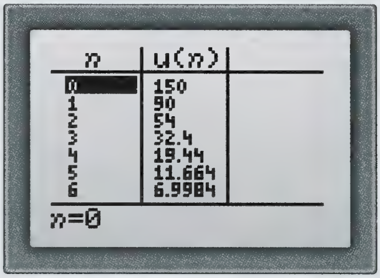
The pollutant level in Lake B will only stabilize if no pollution enters it or if some amount leaves it. Since Lake A is providing pollutant to Lake B continuously and none is leaving, the amount of pollutant in Lake B will continue to rise.

The pollutant level in Lake A will stabilize at 500 t after about 115 years. (**Note:** The number of years will vary depending on the number of decimal places you use.) At this point, 45 t of pollutant will be leaving the lake which matches the amount entering it. (Notice that in a stable state, the amount of pollutant entering the lake will have to match the amount leaving. Since 9% leaves each year, you could solve the equation $45 = 0.09x$ to find the amount of pollutant in the stable state.)

c. Textbook exercise 9 of "Exercises: Extending Your Thinking," p. 273

9. a.

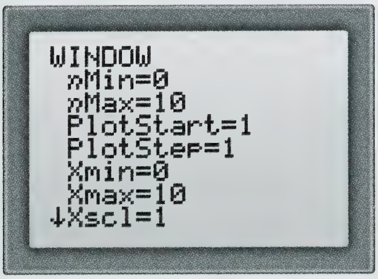


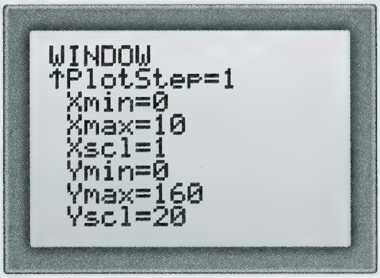


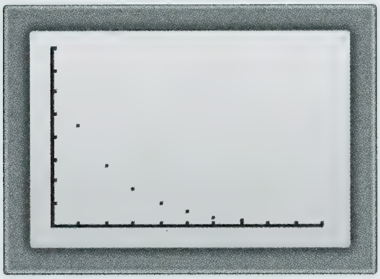
You could set up your own table manually by calculating the amount remaining for each time interval.

Elapsed Time (h)	Amount Remaining (mg)	Calculation
0	150.000	150
1	90.000	150×0.6
2	54.000	150×0.6^2
3	32.400	150×0.6^3
4	19.440	150×0.6^4
5	11.664	150×0.6^5
6	6.998	150×0.6^6

b. Your graph should look like the following.







You may wish to review Utility 33 on pages 363 and 364 of the textbook for information on the meaning of *n*Min, *n*Max, and so on.

c. From the pattern in the table in exercise 9.a., you can obtain the equation $y = 150 \times (0.6)^x$.
You may also enter the data from the table into lists L1 and L2 and use the Exponential Regression feature.

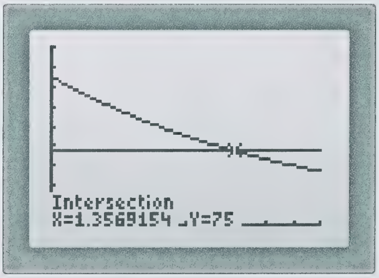
d. Because 60% of the amount from the previous hour remains after each hour, the equation will have powers of 0.6. This leads to the exponential form given in the answer to exercise 9.c.

e. Using the regression equation and your graphing calculator, you can obtain the following table.

Time of Day	Amount Remaining (mg)
8:00 A.M.	150
8:15 A.M.	132.017
8:30 A.M.	116.190
8:45 A.M.	102.260
9:00 A.M.	90
9:15 A.M.	79.210
9:30 A.M.	69.714

Half the amount of penicillin remains sometime between 9:15 A.M. and 9:30 A.M. Therefore, the half-life of penicillin is between 1.25 h and 1.50 h.

Alternately, you could solve the equation $y = 150 \times (0.6)^x$ when y is set to 75. Press MODE, place the cursor over "Func," and press ENTER to be able to enter the equations into the equation editor. Once the equations are entered, use the Intersect feature to find the value of x.



According to the calculator, the half-life is about 1.357 h.

5. **Project Book exercises 1, 2, and 3 of "Getting Started," pp. 116 and 117**

1. a. Answers may vary. A sample spreadsheet is given. Both the numerical and formula versions are shown.

	A	B	C	D	E
1	**Can**	**Radius (cm)**	**Height (cm)**	**Volume (cm^3)**	**Surface Area (cm^2)**
2	1	10	20	6283.185307	1884.955592
3	2	5	10	785.3981634	471.238898
4	3	2.5	5	98.17477042	117.8097245
5	4	1.25	2.5	12.2718463	29.45243113
6	5	0.625	1.25	1.533980788	7.363107782
7	6	0.3125	0.625	0.191747598	1.840776945
8	7	0.15625	0.3125	0.02396845	0.460194236
9	8	0.078125	0.15625	0.002996056	0.115048559
10	9	0.0390625	0.078125	0.000374507	0.02876214
11	10	0.01953125	0.0390625	4.68134E-05	0.007190535

	A	B	C	D	E
1	**Can**	**Radius (cm)**	**Height (cm)**	**Volume (cm^3)**	**Surface Area (cm^2)**
2	1	10	20	=PI()*B2^2*C2	=2*PI()*B2*(B2+C2)
3	=A2+1	=B2/2	=C2/2	=PI()*B3^2*C3	=2*PI()*B3*(B3+C3)
4	=A3+1	=B3/2	=C3/2	=PI()*B4^2*C4	=2*PI()*B4*(B4+C4)
5	=A4+1	=B4/2	=C4/2	=PI()*B5^2*C5	=2*PI()*B5*(B5+C5)
6	=A5+1	=B5/2	=C5/2	=PI()*B6^2*C6	=2*PI()*B6*(B6+C6)
7	=A6+1	=B6/2	=C6/2	=PI()*B7^2*C7	=2*PI()*B7*(B7+C7)
8	=A7+1	=B7/2	=C7/2	=PI()*B8^2*C8	=2*PI()*B8*(B8+C8)
9	=A8+1	=B8/2	=C8/2	=PI()*B9^2*C9	=2*PI()*B9*(B9+C9)
10	=A9+1	=B9/2	=C9/2	=PI()*B10^2*C10	=2*PI()*B10*(B10+C10)
11	=A10+1	=B10/2	=C10/2	=PI()*B11^2*C11	=2*PI()*B11*(B11+C11)

b. The sixth can in the progression has a volume of about $0.1917\ \text{cm}^3$ and a surface area of about $1.8408\ \text{cm}^2$.

c. $r_1 = 10$ and $h_1 = 20$

$$\begin{aligned}\therefore SA &= 2\pi r_1 (r_1 + h_1)\\ &= 2\pi(10)(10+20)\\ &= 2\pi(10)(30)\\ &= 600\pi\end{aligned}$$

$r_2 = \dfrac{10}{2}$ and $h_2 = \dfrac{20}{2}$

$$\begin{aligned}\therefore SA &= 2\pi r_2 (r_2 + h_2)\\ &= 2\pi\left(\frac{10}{2}\right)\left(\frac{10}{2}+\frac{20}{2}\right)\\ &= \frac{20\pi}{2}\left(\frac{30}{2}\right)\\ &= \frac{600\pi}{2^2}\end{aligned}$$

$r_3 = \dfrac{10}{2^2}$ and $h_3 = \dfrac{20}{2^2}$

$$\begin{aligned}\therefore SA &= 2\pi r_3 (r_3 + h_3)\\ &= 2\pi\left(\frac{10}{2^2}\right)\left(\frac{10}{2^2}+\frac{20}{2^2}\right)\\ &= \frac{20\pi}{2^2}\left(\frac{30}{2^2}\right)\\ &= \frac{600\pi}{2^4}\end{aligned}$$

$r_4 = \dfrac{10}{2^3}$ and $h_4 = \dfrac{20}{2^3}$

$$\begin{aligned}\therefore SA &= 2\pi r_4 (r_4 + h_4)\\ &= 2\pi\left(\frac{10}{2^3}\right)\left(\frac{10}{2^3}+\frac{20}{2^3}\right)\\ &= \frac{20\pi}{2^3}\left(\frac{30}{2^3}\right)\\ &= \frac{600\pi}{2^6}\end{aligned}$$

$r_n = \frac{10}{2^{n-1}}$ and $h_n = \frac{20}{2^{n-1}}$

$$\begin{aligned}\therefore\ SA &= 2\pi r_n\left(r_n + h_n\right)\\ &= 2\pi\left(\frac{10}{2^{n-1}}\right)\left(\frac{10}{2^{n-1}} + \frac{20}{2^{n-1}}\right)\\ &= \frac{20\pi}{2^{n-1}}\left(\frac{30}{2^{n-1}}\right)\\ &= \frac{600\pi}{2^{2n-2}}\end{aligned}$$

The expression for surface area is $SA = \frac{600\pi}{2^{2n-2}}$ or $SA = \frac{600\pi}{2^{2(n-1)}}$.

d. $r_1 = 10$ and $h_1 = 20$

$$\begin{aligned}\therefore\ V &= \pi r_1^2 h_1\\ &= \pi(10)^2(20)\\ &= \pi(100)(20)\\ &= 2000\pi\end{aligned}$$

$r_2 = \frac{10}{2}$ and $h_2 = \frac{20}{2}$

$$\begin{aligned}\therefore\ V &= \pi r_2^2 h_2\\ &= \pi\left(\frac{10}{2}\right)^2\left(\frac{20}{2}\right)\\ &= \pi\left(\frac{100}{2^2}\right)\left(\frac{20}{2}\right)\\ &= \frac{2000\pi}{2^3}\end{aligned}$$

$r_3 = \frac{10}{2^2}$ and $h_3 = \frac{20}{2^2}$

$$\begin{aligned}\therefore\ V &= \pi r_3^2 h_3\\ &= \pi\left(\frac{10}{2^2}\right)^2\left(\frac{20}{2^2}\right)\\ &= \pi\left(\frac{100}{2^4}\right)\left(\frac{20}{2^2}\right)\\ &= \frac{2000\pi}{2^6}\end{aligned}$$

$r_4 = \dfrac{10}{2^3}$ and $h_4 = \dfrac{20}{2^3}$

$$\therefore V = \pi r_4^2 h_4$$
$$= \pi\left(\frac{10}{2^3}\right)^2\left(\frac{20}{2^3}\right)$$
$$= \pi\left(\frac{100}{2^6}\right)\left(\frac{20}{2^3}\right)$$
$$= \frac{2000\pi}{2^9}$$

$r_n = \dfrac{10}{2^{n-1}}$ and $h_n = \dfrac{20}{2^{n-1}}$

$$\therefore V = \pi r_n^2 h_n$$
$$= \pi\left(\frac{10}{2^{n-1}}\right)^2\left(\frac{20}{2^{n-1}}\right)$$
$$= \pi\left(\frac{100}{2^{2(n-1)}}\right)\left(\frac{20}{2^{n-1}}\right)$$
$$= \frac{2000\pi}{2^{3n-3}}$$

The expression for volume is $V = \dfrac{2000\pi}{2^{3n-3}}$ or $V = \dfrac{2000\pi}{2^{3(n-1)}}$.

2. a.

$$355 = \pi r^2 h$$
$$\frac{355}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$
$$h = \frac{355}{\pi r^2}$$

b.

$$A = 2\pi r^2 + 2\pi rh$$
$$A = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right)$$
$$A = 2\pi r^2 + 2\left(\frac{355}{r}\right)$$
$$A = 2\pi r^2 + \frac{710}{r}$$

c. and d.

	A	B	C	D
1	Radius (cm)	Height (cm)	Surface Area (cm²)	Cost ($)
2	1	113.0000096	716.2831853	0.36
3	2	28.2500024	380.1327412	0.19
4	3	12.55555662	293.2153344	0.15
5	4	7.0625006	278.0309649	0.14
6	5	4.520000384	299.0796327	0.15
7	6	3.138889155	344.5280044	0.17
8	7	2.306122645	409.3046515	0.20
9	8	1.76562515	490.8738597	0.25
10	9	1.395061847	587.8268988	0.29
11	10	1.130000096	699.3185307	0.35

	A	B	C	D
1	**Radius (cm)**	**Height (cm)**	**Surface Area (cm²)**	**Cost ($)**
2	1	=355/(PI()*A2^2)	=2*PI()*A2^2+710/A2	=C2*0.0005
3	=A2+1	=355/(PI()*A3^2)	=2*PI()*A3^2+710/A3	=C3*0.0005
4	=A3+1	=355/(PI()*A4^2)	=2*PI()*A4^2+710/A4	=C4*0.0005
5	=A4+1	=355/(PI()*A5^2)	=2*PI()*A5^2+710/A5	=C5*0.0005
6	=A5+1	=355/(PI()*A6^2)	=2*PI()*A6^2+710/A6	=C6*0.0005
7	=A6+1	=355/(PI()*A7^2)	=2*PI()*A7^2+710/A7	=C7*0.0005
8	=A7+1	=355/(PI()*A8^2)	=2*PI()*A8^2+710/A8	=C8*0.0005
9	=A8+1	=355/(PI()*A9^2)	=2*PI()*A9^2+710/A9	=C9*0.0005
10	=A9+1	=355/(PI()*A10^2)	=2*PI()*A10^2+710/A10	=C10*0.0005
11	=A10+1	=355/(PI()*A11^2)	=2*PI()*A11^2+710/A11	=C11*0.0005

3. a. The least-cost size from exercise 2 has a radius of 4 cm.

b. Actual containers have a radius of about 3.25 cm and a height of about 12 cm. So, the values from exercise 2 give a shorter can that is larger in radius.

6. Since soft-drink containers of 355 mL are meant to be held in a hand while being consumed, the container has to be able to be held in most hands. This gives bounds to the height and radius. The human factors would be hand size and the length of the arm that would position the can for drinking.

7. The larger the surface area of a container in a refrigerator, the quicker the contents will cool (all other things being equal). Thus, a shape with a larger surface area will allow the contents to cool more quickly.

8. Textbook exercise "Communicating the Ideas," p. 273

Each term in the sequence here is obtained from its immediate predecessor by multiplying by a fixed number (the common ratio). For example, suppose the initial term is 6 and the common ratio is 0.1. The first calculated term is $6\times(0.1)^1$; the second calculated term is $6\times(0.1)^2$; the third calculated term is $6\times(0.1)^3$; and so on. The associated function for this sequence, then, is $f(x)=6\times(0.1)^x$. This is an exponential function, so the related graph is exponential in form.

Activity 3: Introduction to Fractals

1. Textbook exercises 1 to 9 of "Investigation 1: Create a 3-Dimensional Fractal," pp. 276 and 277

1. to 5. Refer to the pictures on pages 276 and 277 of the textbook.

Activity 3 (continued)

6. and 7.

	Number of Additional Boxes	Length (cm)	Width (cm)	Height (cm)	Volume of One New Box (cm^3)	Volume of All New Boxes (cm^3)	Total Volume of All Boxes (cm^3)
Original	1	10.8	7.0	7.0	529.2	529.2	529.2
Iteration 1	2	5.4	3.5	3.5	66.15	132.3	661.5
Iteration 2	4	2.7	1.75	1.75	8.268 75	33.075	694.575
Iteration 3	8	1.35	0.875	0.875	1.033 593 75	8.268 75	702.843 75
Iteration 4	16	0.675	0.4375	0.4375	0.129 199 219	2.067 187 5	704.910 937 5

Here are the calculations for Iteration 3.

$$\begin{aligned}\text{Length} &= \frac{2.7}{2} \\ &= 1.35\text{ cm}\end{aligned} \qquad \begin{aligned}\text{Width} = \text{Height} &= \frac{1.75}{2} \\ &= 0.875\text{ cm}\end{aligned}$$

$$\begin{aligned}\text{Volume of one new box} &= \text{Length} \times \text{Width} \times \text{Height} \\ &= 1.35 \times 0.875 \times 0.875 \\ &\doteq 1.033\ 593\ 75\end{aligned}$$

$$\begin{aligned}\text{Volume of all new boxes} &= \text{Volume of one new box} \times \text{Number of additional boxes} \\ &\doteq 1.033\ 593\ 75 \times 8 \\ &\doteq 8.268\ 75\end{aligned}$$

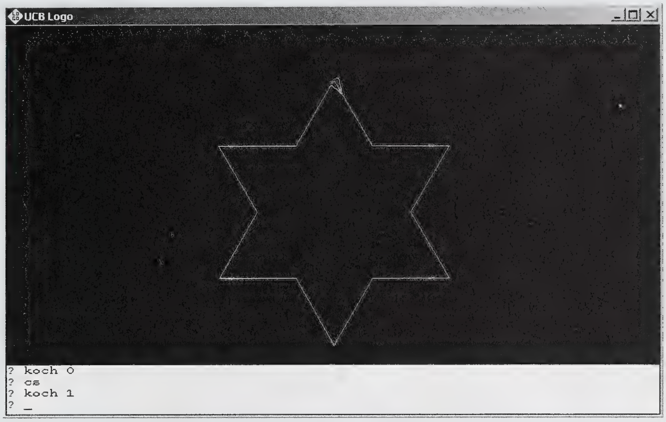
$$\begin{aligned}\text{Total volume of all boxes} &= \text{Total volume from iteration 2} + \text{Volume of all new boxes} \\ &\doteq 694.575 + 8.268\ 75 \\ &\doteq 702.843\ 75\end{aligned}$$

8. The sequence $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$ describes the number of boxes at each stage. The original would be interpreted as iteration 0. At stage 10, you would have completed iteration 9, in which 2^9, or 512, additional boxes are made.

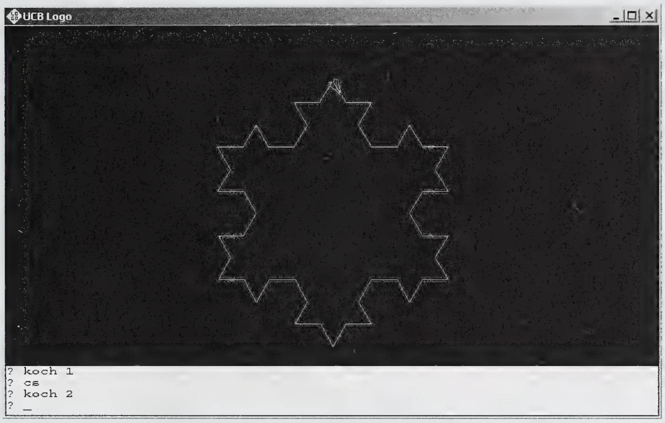
9. The sequence $\frac{529.2}{8^0}, \frac{529.2}{8^1}, \frac{529.2}{8^2}, \frac{529.2}{8^3}, \ldots$ describes the volume of one new box at each stage. At stage 12, you would have completed iteration 11, in which the volume of one new box would be $\frac{529.2}{8^{11}}\text{ cm}^3$, or $0.000\ 000\ 061\ 606\ 988\ 31\text{ cm}^3$.

2. Your sketches of the Koch snowflake should look like the following.

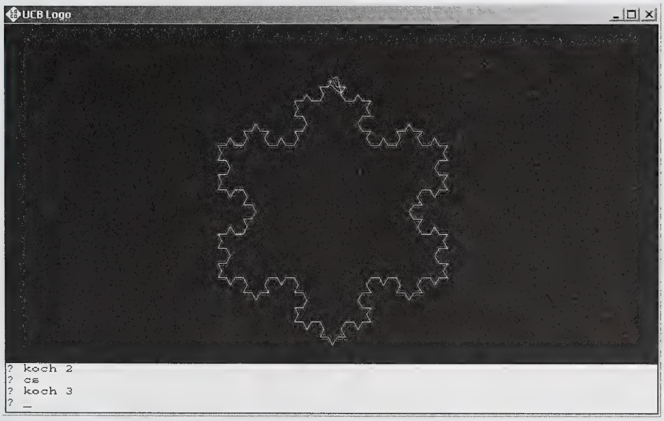
Iteration 1:



Iteration 2:



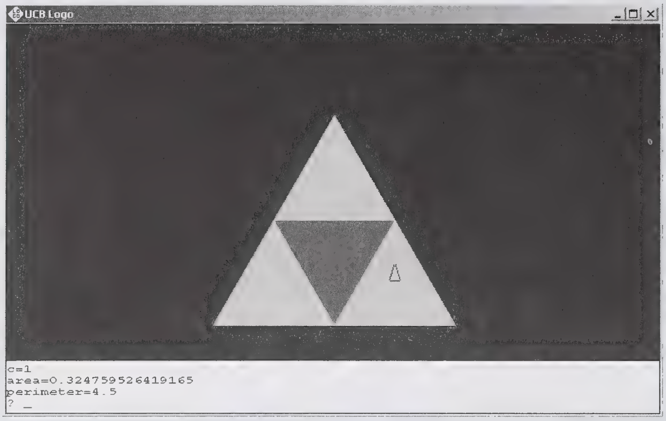
Iteration 3:



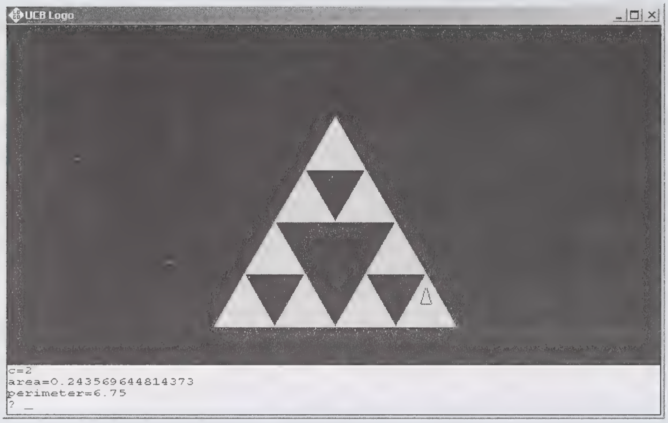
Activity 3 (continued)

3. Your sketches of the Sierpinski gasket should look like the following.

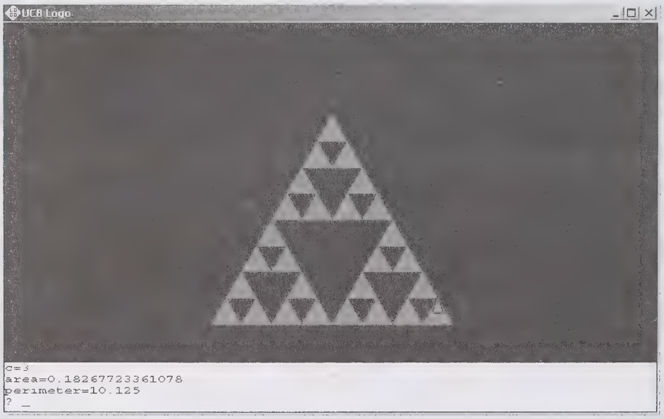
Iteration 1:



Iteration 2:



Iteration 3:



4. a. **Textbook exercise 1 of "Discussing the Ideas," p. 281**

1. Answers will vary. A sample answer is given.

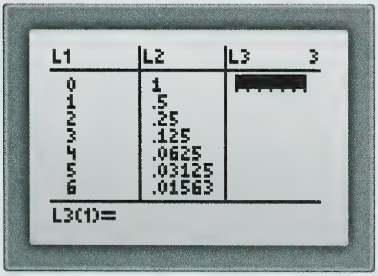
   Yes, other three-dimensional fractals can be created by cutting and folding. For example, you could use practically the same method except fold out the sides of the paper instead of folding in. You will end up with a different three-dimensional fractal.

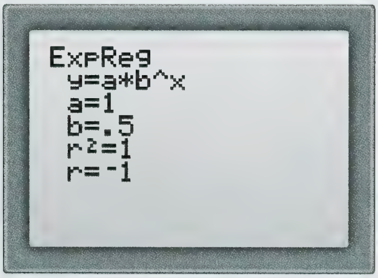
b. There are several sequences in the Sierpinski gasket: one is the lengths of the sides, another is the perimeter, and another is the area. These are shown in the following table.

Iteration	Length of Side of Each Triangle	Area of Gasket	Perimeter of Gasket
0	1.0	0.433.013	3.0
1	0.5	0.324 760	4.5
2	0.25	0.243 570	6.75
3	0.125	0.182 677	10.125
4	0.0625	0.137 008	15.1875
5	0.031 25	0.102 756	22.781 25
6	0.015 625	0.077 067	34.171 875

If you plot the values from one of the columns against the iteration number, you will see a typical exponential form.

Length of Side of Each Triangle

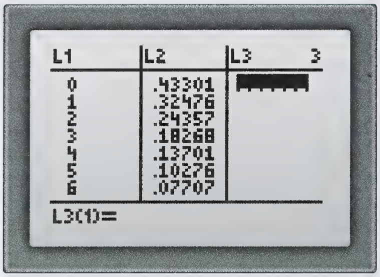


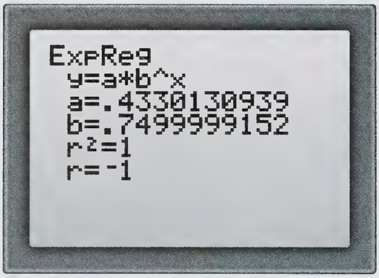


For the length of each side, the sequence is as follows:

$$1\times\left(\frac{1}{2}\right)^0, 1\times\left(\frac{1}{2}\right)^1, 1\times\left(\frac{1}{2}\right)^2, 1\times\left(\frac{1}{2}\right)^3, \ldots, 1\times\left(\frac{1}{2}\right)^n, \ldots$$

Area of Gasket

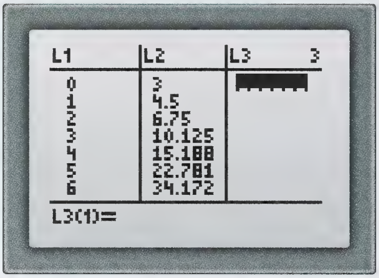


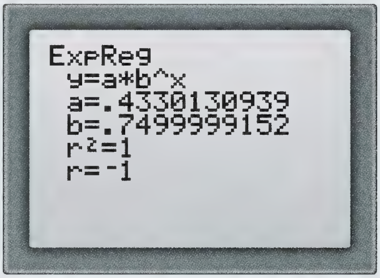


For the area of the Sierpinski gasket, the sequence is as follows:

$$0.433\times\left(\frac{3}{4}\right)^0, 0.433\times\left(\frac{3}{4}\right)^1, 0.433\times\left(\frac{3}{4}\right)^2, 0.433\times\left(\frac{3}{4}\right)^3, \ldots, 0.433\times\left(\frac{3}{4}\right)^n, \ldots$$

Perimeter of Gasket





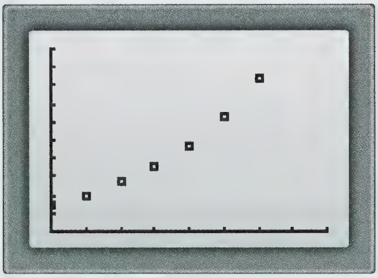
For the perimeter of the Sierpinski gasket, the sequence is as follows:

$$3\times\left(\frac{3}{2}\right)^0, 3\times\left(\frac{3}{2}\right)^1, 3\times\left(\frac{3}{2}\right)^2, 3\times\left(\frac{3}{2}\right)^3, \ldots, 3\times\left(\frac{3}{2}\right)^n, \ldots$$

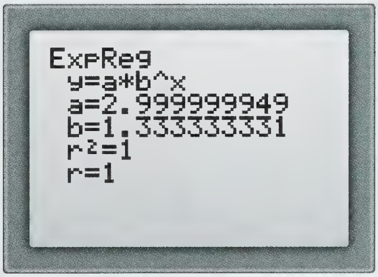
c. The following table shows perimeter of the Koch snowflake after six iterations.

Iteration	[illegible]
0	3.0
1	4.0
2	5.333 333
3	7.111 111
4	9.481 481
5	12.641 975
6	16.855 967

Plot the points on your graphing calculator.



The pattern looks exponential. Trying an exponential regression gives the following sequence for the perimeter of the Koch snowflake.



$$3\times\left(\frac{4}{3}\right)^0, 3\times\left(\frac{4}{3}\right)^1, 3\times\left(\frac{4}{3}\right)^2, 3\times\left(\frac{4}{3}\right)^3, \ldots, 3\times\left(\frac{4}{3}\right)^n, \ldots$$

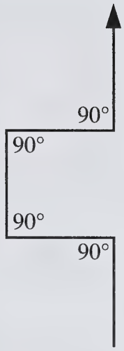
This shows that the perimeter gets larger with each iteration. In a sense, the Koch snowflake (with infinite iterations) will have an infinite perimeter.

d. Computers and calculators are very good at repeating calculations. This makes generating self-similar patterns a matter of telling the computer what to do, how many times to do it, and then allowing it to get to work on it.

Activity 3 (continued)

5. Answers may vary. A sample answer is given.

To generate the diagram given, the steps needed are as follows:



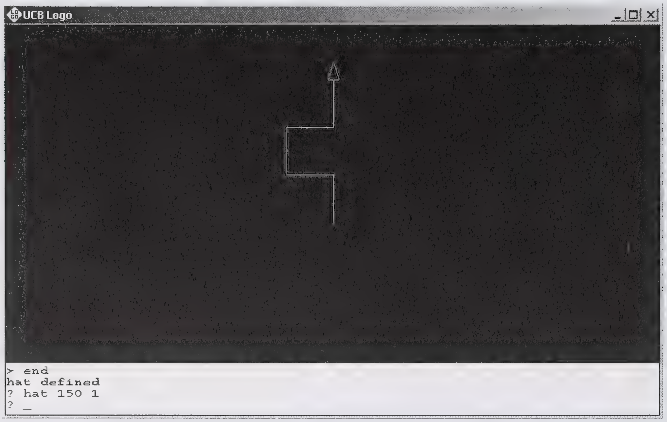
- Go forward one-third the length
- Turn 90 degrees left
- Go forward one-third the length
- Turn 90 degrees right
- Go forward one-third the length
- Turn 90 degrees right
- Go forward one-third the length
- Turn 90 degrees left
- Go forward one-third the length

Generalizing these steps gives the following procedure. It is very much like Bumpy line procedure except for the number of repetitions and the turn angles.

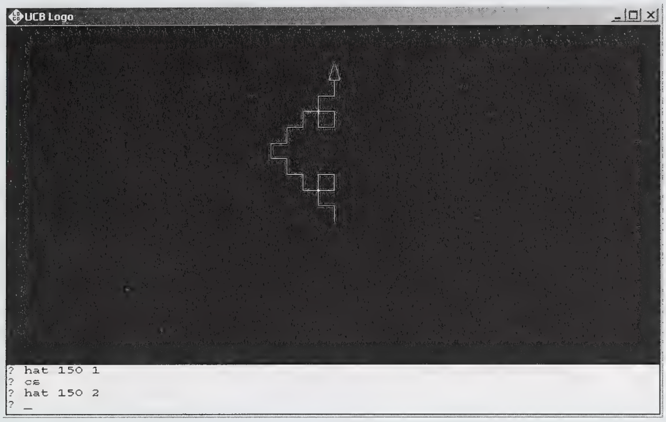
```
? to hat :s :b
> if (:b = 0) [fd :s stop]
> hat :s/3 :b-1 lt 90
> hat :s/3 :b-1 rt 90
> hat :s/3 :b-1 rt 90
> hat :s/3 :b-1 lt 90
> hat :s/3 :b-1
> end
```

The first three iterations of this procedure look like the following.

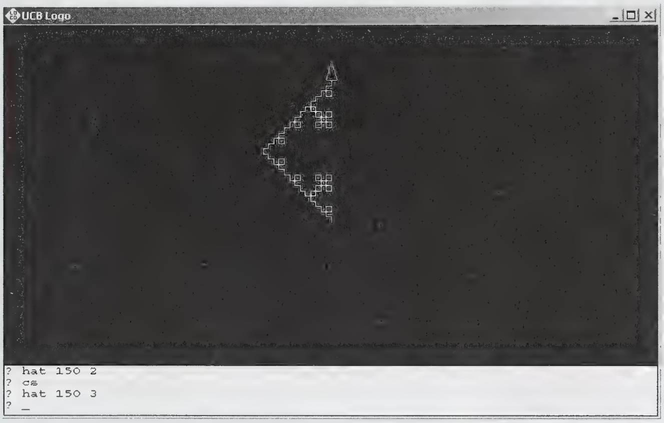
Iteration 1:



Iteration 2:



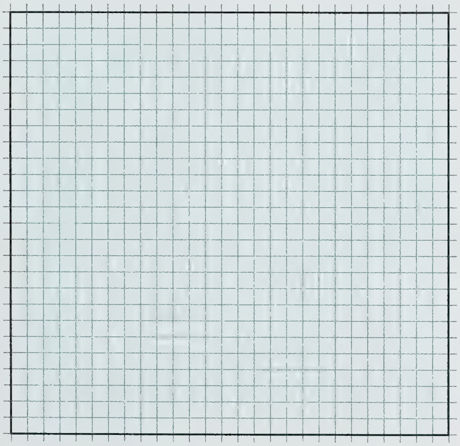
Iteration 3:



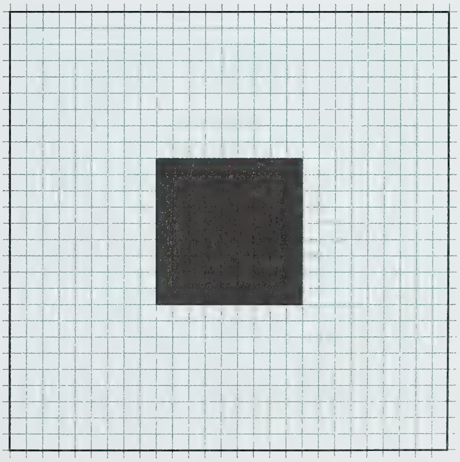
6. Textbook exercise 3 of "Exercises: Checking Your Skills," p. 282

3. You can make these designs using *Logo* and the program *sr.lg*.

a.

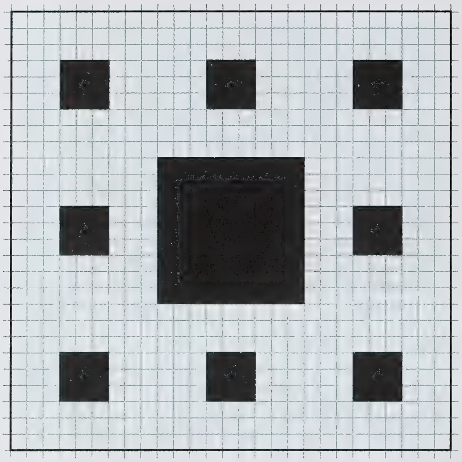


b.

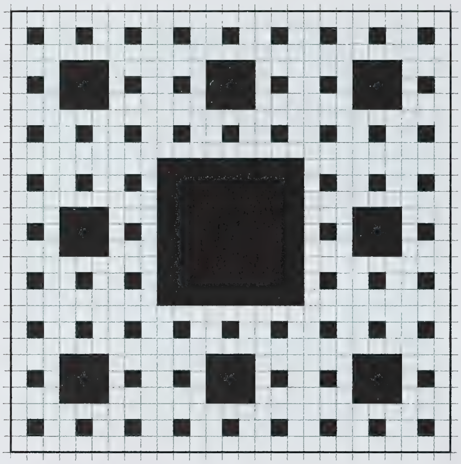


Activity 3 (continued)

c. and d.



e.



7. Textbook exercise "Communicating the Ideas," p. 282

Self-similarity of an image means that if a part of the image is enlarged, it will look like the whole image. In exercise 3 on page 282 of the textbook, the images created were self-similar. If the upper left corner of the image from exercise 3.e. were enlarged, it would look just like the image in exercise 3.d.

Iteration means to repeat the step or follow the directions again.

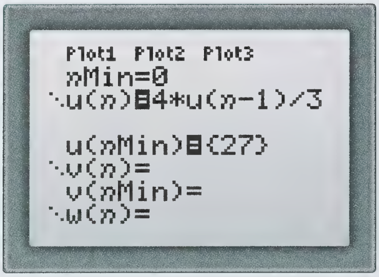
Activity 4: Some Properties of Fractals

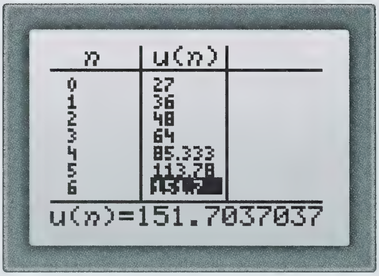
1. Textbook exercises 2, 3, and 4 of Part A of "Investigation 1: Properties of a Koch Snowflake," p. 284

2. Below the values in the table are the calculations used to obtain the values.

Iteration	[illegible]	Length of Each Side (cm)	[illegible]
Original	3	9	27
1	12 $=3\times 4$ $=3\times 4^1$	3 $=9\div 3$ $=9\div 3^1$	36 $=12\times 3$ $=\left(3\times 4^1\right)\times\left(9\div 3^1\right)$ $=4^1\times 3^2$
2	48 $=3\times 16$ $=3\times 4^2$	1 $=9\div 9$ $=9\div 3^2$	48 $=48\times 1$ $=\left(3\times 4^2\right)\times\left(9\div 3^2\right)$ $=4^2\times 3^1$
3	192 $=3\times 64$ $=3\times 4^3$	$\frac{1}{3}$ $=9\div 27$ $=9\div 3^3$	64 $=192\times\frac{1}{3}$ $=\left(3\times 4^3\right)\times\left(9\div 3^3\right)$ $=4^3\times 3^0$
4	768 $=3\times 256$ $=3\times 4^4$	$\frac{1}{9}$ $=9\div 81$ $=9\div 3^4$	$85\frac{1}{3}$ $=768\times\frac{1}{9}$ $=\left(3\times 4^4\right)\times\left(9\div 3^4\right)$ $=4^4\times 3^{-1}$

3.





The total perimeter after six iterations is approximately 151.70 cm.

$$\begin{aligned}(3\times 4^6)\times(9\div 3^6) &= 3^1\times 4^6\times 3^2\div 3^6\\ &= 4^6\times 3^{1+2-6}\\ &= 4^6\times 3^{-3}\\ &= \frac{4096}{27}\\ &\doteq 151.703\,703\,7\end{aligned}$$

4. The perimeter will increase without limit as the number of iterations increases indefinitely. The perimeter at the nth iteration is given by the expression $(3\times 4^n)\times(9\div 3^n)$, or $27\left(\frac{4}{3}\right)^n$. This shows that the perimeter increases by a factor of $\frac{4}{3}$ at each iteration. You may wish to try substituting values like 10, 50, and 100 for n and finding the perimeter.

b. Textbook exercises 1 to 5 of Part B of "Investigation 1: Properties of a Koch Snowflake," pp. 284 and 285

1. The original triangle has an area of 81 square units.

2. Below the values in the table are the calculations used to obtain the values. An additional column has been added to the table to make it easy to see the number of new triangles. The number of new triangles is the same as the number of sides in the previous iteration, since each side is replaced with a bumpy side with a new triangle on it.

Iteration	Area of One New Triangle	Number of New Triangles Added	Number of Sides	Total Area Added	Total Area
Original	81	1	3	81	81
1	9 $=81\div 9$ $=81\div 9^{1}$	3	12 $=3\times 4$ $=3\times 4^{1}$	27 $=9\times 3$ $=(81\div 9^{1})\times(3^{1}\times 4^{0})$	108 $=81+27$
2	1 $=81\div 81$ $=81\div 9^{2}$	12	48 $=3\times 16$ $=3\times 4^{2}$	12 $=1\times 12$ $=(81\div 9^{2})\times(3\times 4^{1})$ $=3^{1}\times 4^{1}$	120 $=108+12$
3					
4					

3. The area of a new triangle decreases by a factor of 9 at each iteration.

Activity 4 (continued)

4. Below the values in the table are the calculations used to obtain the values. An additional column has been added to the table to make it easy to see the number of new triangles.

Iteration	Area of One New Triangle	Number of New Triangles Added	Number of Sides	Total Area Added	Total Area
Original	81	1	3	81	81
1	9 $=81\div 9$ $=81\div 9^{1}$	3	12 $=3\times 4$ $=3\times 4^{1}$	27 $=9\times 3$ $=\left(81\div 9^{1}\right)\times\left(3^{1}\times 4^{0}\right)$	108 $=81+27$
2	1 $=81\div 81$ $=81\div 9^{2}$	12 $=3\times 4$ $=3\times 4^{1}$	48 $=3\times 16$ $=3\times 4^{2}$	12 $=1\times 12$ $=\left(81\div 9^{2}\right)\times\left(3\times 4^{1}\right)$ $=3^{1}\times 4^{1}$	120 $=108+12$
3	$\frac{1}{9}$ $=81\div 729$ $=81\div 9^{3}$	48 $=3\times 16$ $=3\times 4^{2}$	192 $=3\times 64$ $=3\times 4^{3}$	$\frac{48}{9}$ or $5.\overline{3}$ $=\frac{1}{9}\times 48$ $=\left(81\div 9^{3}\right)\times\left(3\times 4^{2}\right)$ $=3^{-1}\times 4^{2}$	$125.\overline{3}$ $=120+5.\overline{3}$
4	$\frac{1}{81}$ $=81\div 6561$ $=81\div 9^{4}$	192 $=3\times 64$ $=3\times 4^{3}$	768 $=3\times 256$ $=3\times 4^{4}$	$\frac{192}{81}$ or $2.\overline{370}$ $=\frac{1}{81}\times 192$ $=\left(81\div 9^{4}\right)\times\left(3\times 4^{3}\right)$ $=3^{-3}\times 4^{3}$	$127.\overline{703}$ $=125.\overline{3}+2.\overline{370}$

5. The area is increasing with each iteration. However, the amount the area is increasing with each iteration is actually decreasing. In fact, the total area will approach about 130 square units as the number of iterations increases indefinitely.

The total area can be determined as follows:

$$\begin{aligned}\text{Total area} &= 81+27+12+5.\bar{3}+2.\overline{370}+\ldots\\ &= 81+27+\left(3^{1}\times 4^{1}\right)+\left(3^{-1}\times 4^{2}\right)+\left(3^{-3}\times 4^{3}\right)+\left(3^{-5}\times 4^{4}\right)+\left(3^{-7}\times 4^{5}\right)+\ldots\\ &= 108+12+\frac{4^{2}}{3}+\frac{4^{3}}{3^{3}}+\frac{4^{4}}{3^{5}}+\frac{4^{5}}{3^{7}}+\ldots\\ &= 108+12\times\left(1+\frac{4}{9}+\frac{4^{2}}{9^{2}}+\frac{4^{3}}{9^{3}}+\frac{4^{4}}{9^{4}}+\frac{4^{5}}{9^{5}}+\ldots\right)\end{aligned}$$

If you could find the sum inside the parentheses, the total area would be known. There is a trick that can be used to find sums like this. Let the sum of the first *n* terms be called SUM.

Multiply SUM by the common ratio.

$$\text{SUM}=1+\frac{4}{9}+\frac{4^{2}}{9^{2}}+\frac{4^{3}}{9^{3}}+\frac{4^{4}}{9^{4}}+\frac{4^{5}}{9^{5}}+\ldots+\frac{4^{n-1}}{9^{n-1}}$$

$$\frac{4}{9}(\text{SUM})=\frac{4}{9}+\frac{4^{2}}{9^{2}}+\frac{4^{3}}{9^{3}}+\frac{4^{4}}{9^{4}}+\frac{4^{5}}{9^{5}}+\ldots+\frac{4^{n-1}}{9^{n-1}}+\frac{4^{n}}{9^{n}}$$

If you subtract the second equation from the first one, you get the following:

$$\text{SUM}-\frac{4}{9}(\text{SUM})=1-\frac{4^{n}}{9^{n}}$$

The part after the 1 in the first sum is the same as the part before $\frac{4^n}{9^n}$ in the second sum.

This can be rearranged to give the following equation:

$$\text{SUM}\left(1-\frac{4}{9}\right)=1-\frac{4^{n}}{9^{n}}$$

With more rearrangement you get the following:

$$\text{SUM}=\frac{1-\left(\frac{4}{9}\right)^{n}}{\frac{5}{9}}$$

As *n* becomes larger and larger, the value of $\left(\frac{4}{9}\right)^{n}$ becomes smaller and smaller until the point is reached where it can be considered to be 0.

$$\begin{aligned}\therefore\ \text{Total area} &= 108 + 12 \times \left(\frac{1-0}{\frac{5}{9}} \right) \\ &= 108 + 12 \times \left(\frac{1}{\frac{5}{9}} \right) \\ &= 108 + 12 \times \frac{9}{5} \\ &= 129.6\end{aligned}$$

2. a. **Textbook exercises 1, 2, and 3 of Part A of "Investigation 2: Some Properties of a Sierpinski Gasket," pp. 285 and 286**

1. Below the values in the table are the calculations used to obtain the values.

Iteration	Number of Shaded Triangles	Length of Each Side	Total Perimeter
Original	1	16	48
1	3	8 $= 16 \div 2^1$	72 $= 8 \times 3 \times 3$
2	9 $= 1 \times 3^2$	4 $= 16 \div 2^2$	108 $= 4 \times 9 \times 3$
3			
4			

2. The number of shaded triangles forms a geometric sequence. You could use the rule $S_n = 3 \times S_{n-1}$, where n is the number of iterations. Also, $t_n = 3^n$ works, but it is not a recursive rule.

3. Below the values in the table are the calculations used to obtain the values.

Iteration	Number of Shaded Triangles	Length of Each Side	Total Perimeter
Original	1	16	48
1	3	8 $= 16 \div 2^1$	72 $= 8 \times 3 \times 3$
2	9 $= 3^2$	4 $= 16 \div 2^2$	108 $= 4 \times 9 \times 3$
3	27 $= 3^3$	2 $= 16 \div 2^3$	162 $= 2 \times 27 \times 3$
4	81 $= 3^4$	1 $= 16 \div 2^4$	243 $= 1 \times 81 \times 3$

b. Textbook exercises 1 to 5 of Part B of "Investigation 2: Some Properties of a Sierpinski Gasket," p. 286

1. The shaded triangles in iteration 1 are one-quarter the area of the original triangle. The lengths of the sides are halved, which makes the area one-quarter. Also, you can see that there are four triangles making up iteration 1 and that they are all the same size; however, they take up the same space as the original triangle.

Activity 4 (continued)

2. and 3. Below the values in the table are the calculations used to obtain the values.

Iteration	Number of Smallest Triangles	Area of Smallest Triangle	Total Area
Original	1	64	64
1	3 $= 1 \times 3^1$	16 $= 64 \div 4^1$	48 $= 3 \times 16$
2	9 $= 1 \times 3^2$	4 $= 64 \div 4^2$	36 $= 9 \times 4$
3			
4			

4. Below the values in the table are the calculations used to obtain the values.

Iteration	Number of Smallest Triangles	Area of Smallest Triangle	Total Area
Original	1	64	64
1	3 $= 3^1$	16 $= 64 \div 4^1$	48 $= 3 \times 16$
2	9 $= 3^2$	4 $= 64 \div 4^2$	36 $= 9 \times 4$
3	27 $= 3^3$	1 $= 64 \div 4^3$	27 $= 27 \times 1$
4	81 $= 3^4$	0.25 $= 64 \div 4^4$	20.25 $= 81 \times 0.25$

5. The area after n iterations is the product of the number of triangles and the area of one triangle. The number of triangles after n iterations is given by 3^n, and the area of one of the triangles is given by $64 \div 4^n$. These give the following expression for the area of the Sierpinski gasket after n iterations:

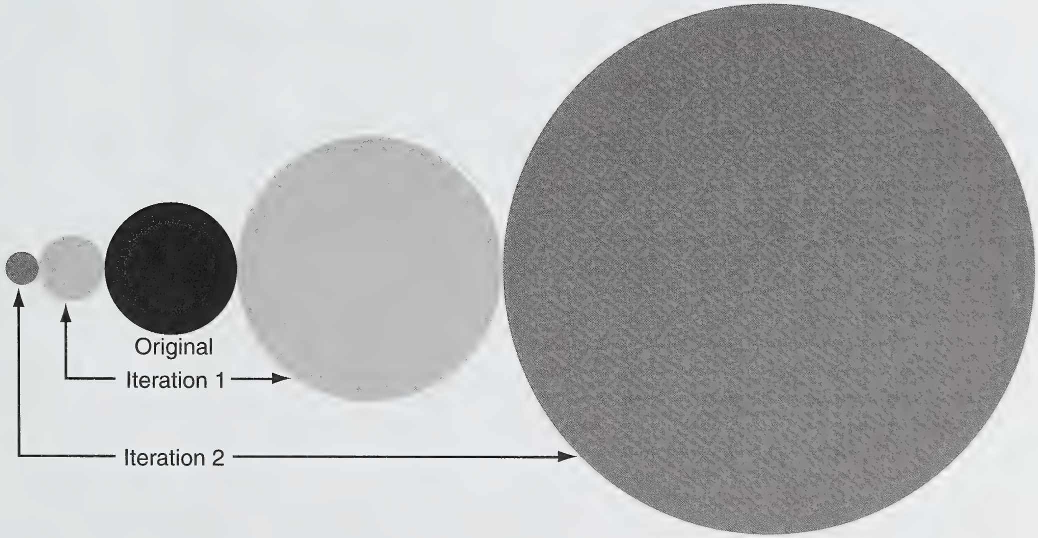
$$\begin{aligned} SA_n &= 3^n \times \left(64 \div 4^n\right) \\ &= 64 \times 3^n \div 4^n \\ &= 64 \times \left(\frac{3}{4}\right)^n \end{aligned}$$

3. a. **Textbook exercises 1, 2, and 3 of "Discussing the Ideas," p. 288**

1. The area of a fractal does not always approach 0 as the number of iterations increases indefinitely. Consider the area of the Koch snowflake. The area increases at each iteration, thus cannot approach 0.

2. The perimeter of many fractals increases indefinitely as the number of iterations increases indefinitely. The Koch snowflake and the Sierpinski gasket are two examples.

3. The area of a fractal can increase indefinitely. Consider a fractal built at each step by adding a circle with double the radius to the right of the current iteration and a second circle with half the radius to the left of the current iteration. The first two iterations are shown in the following diagram.



Activity 4 (continued)

b. Textbook exercises 1.a., 1.b., 2, 3.a., 3.b., and 4 of "Exercises: Checking Your Skills," pp. 289 to 291

1. a. and b. Below the values in the table are the calculations used to obtain the values.

Iteration	Length of Each Segment	Number of Segments	Total Length
Original	243	1	243
1	81 $= 243 \div 3^1$	5 $= 5^1$	405 $= 5 \times 81$
2	27 $= 243 \div 3^2$	25 $= 5^2$	675 $= 25 \times 27$
3	9 $= 243 \div 3^3$	125 $= 5^3$	1125 $= 125 \times 9$
4	3 $= 243 \div 3^4$	625 $= 5^4$	1875 $= 625 \times 3$

2. a. Below the values in the table are the calculations used to obtain the values.

Iteration	Side Length of New Square	Area of a New Square	Number of New Squares	Total Area of New Squares	Total Unshaded Area
1	9 $= 27 \div 3^1$	81 $= 9^2$	1	81	648
2	3 $= 27 \div 3^2$	9 $= 3^2$	8 $= 8^1$	72 $= 9 \times 8$	576 $= 648 - 72$
3	1 $= 27 \div 3^3$	1 $= 1^2$	64 $= 8^2$	64 $= 1 \times 64$	512 $= 576 - 64$
4	$\frac{1}{3}$ $= 27 \div 3^4$	$\frac{1}{9}$ $= \left(\frac{1}{3}\right)^2$	512 $= 8^3$	$56\frac{8}{9}$ $= \frac{1}{9} \times 512$	$455\frac{1}{9}$ $= 512 - 56\frac{8}{9}$

b. The total unshaded area approaches 0 as the number of iterations increases indefinitely. So the total shaded area approaches the area of the original square, which is 729 square units. The following shows how this may be calculated, but it is not a required part of this answer.

The shaded area for n iterations is given by the following sum:

$$81 + 72 + 64 + 56\frac{8}{9} + \ldots + 81\left(\frac{8}{9}\right)^{n-1}$$

$$= 81\left[1 + \left(\frac{8}{9}\right)^1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^3 + \ldots + \left(\frac{8}{9}\right)^{n-1}\right]$$

Using the trick shown earlier allows you to find what this sum will be for any value of n. Let SUM represent the part inside the parentheses.

$$\text{SUM} = 1 + \left(\frac{8}{9}\right)^1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^3 + \ldots + \left(\frac{8}{9}\right)^{n-1}$$

$$\frac{8}{9}(\text{SUM}) = \left(\frac{8}{9}\right)^1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^3 + \ldots + \left(\frac{8}{9}\right)^{n-1} + \left(\frac{8}{9}\right)^n$$

If you subtract the second equation from the first, you will be able to find a closed form for the sum.

$$\text{SUM} - \frac{8}{9}(\text{SUM}) = 1 - \left(\frac{8}{9}\right)^n$$

$$\text{SUM}\left(1 - \frac{8}{9}\right) = 1 - \left(\frac{8}{9}\right)^n$$

$$\text{SUM}\left(\frac{1}{9}\right) = 1 - \left(\frac{8}{9}\right)^n$$

$$\text{SUM} = 9\left[1 - \left(\frac{8}{9}\right)^n\right]$$

As n becomes larger and larger, SUM becomes closer and closer to 9, since $\left(\frac{8}{9}\right)^n$ becomes closer and closer to 0. That means the shaded area will become closer and closer to $81 \times 9 = 729$, which is the total area of the original square. This leaves an unshaded area that approaches 0.

3. **a.** Below the values in the table are the calculations used to obtain the values.

Iteration	Number of New Circles	Radius of One New Circle	Increase in Circumference	Total Circumference of All Circles
Original	1	2	0	4π
1	2 $= 2^1$	1 $= 2 \div 2^1$	4π $2 \times (2\pi r) = 4\pi \times 1$	8π
2	4 $= 2^2$	$\frac{1}{2}$ $= 2 \div 2^2$	4π $4 \times (2\pi r) = 8\pi \times \frac{1}{2}$	12π
3	8 $= 2^3$	$\frac{1}{4}$ $= 2 \div 2^3$	4π $8 \times (2\pi r) = 16\pi \times \frac{1}{4}$	16π
4	16 $= 2^4$	$\frac{1}{8}$ $= 2 \div 2^4$	4π $16 \times (2\pi r) = 32\pi \times \frac{1}{8}$	20π

b. The total circumference will increase without bound as the number of iterations increases indefinitely.

4. The following table shows information about the first four spheres in the sequence.

Sphere	Radius (cm)	Surface Area (cm^2)	Volume (cm^3)
1	64 $= 2^6$	$16\,384\,\pi$ $= 4\pi \times 64^2$	$\frac{1\,048\,576}{3}\pi$ $= \frac{4}{3}\pi \times 64^3$ $= \frac{2^{20}}{3}\pi$
2	32 $= 2^5$ $= 64 \div 2^1$	$4096\,\pi$ $= 4\pi \times 32^2$ $= 4\pi \times \left(\frac{64}{2}\right)^2$	$\frac{131\,072}{3}\pi$ $= \frac{4}{3}\pi \times 32^3$ $= \frac{2^{17}}{3}\pi$
3	16 $= 2^4$ $= 64 \div 2^2$	$1024\,\pi$ $= 4\pi \times 16^2$ $= 4\pi \times \left(\frac{64}{2^2}\right)^2$	$\frac{16\,384}{3}\pi$ $= \frac{4}{3}\pi \times 16^3$ $= \frac{2^{14}}{3}\pi$
4	8 $= 2^3$ $= 64 \div 2^3$	$256\,\pi$ $= 4\pi \times 8^2$ $= 4\pi \times \left(\frac{64}{2^3}\right)^2$	$\frac{2048}{3}\pi$ $= \frac{4}{3}\pi \times 8^3$ $= \frac{2^{11}}{3}\pi$

a. The surface area of a sphere is calculated using the formula $SA = 4\pi r^2$. The surface areas of the spheres give the following sequence:

$$4\pi\times 64^2, 4\pi\times 32^2, 4\pi\times 16^2, 4\pi\times 8^2, 4\pi\times 4^2, 4\pi\times 2^2, 4\pi\times 1^2, \ldots$$

The seventh sphere will have a surface area of $4\pi\times 1^2\ \text{cm}^2$, or $4\pi\ \text{cm}^2$. The *n*th sphere will have a surface area (in square centimetres) of

$$\begin{aligned}4\pi\times\left(\frac{64}{2^{n-1}}\right)^2 &= 4\pi\times\left(\frac{2^6}{2^{n-1}}\right)^2\\ &= 2^2\,\pi\times\left(\frac{2^{12}}{2^{2n-2}}\right)\\ &= \pi\times\left(\frac{2^{14}}{2^{2n-2}}\right)\\ &= \pi\times 2^{14-(2n-2)}\\ &= \pi\times 2^{16-2n}\end{aligned}$$

b. The volume of a sphere is calculated using the formula $V = \frac{4}{3}\pi r^3$. (**Note:** The formula given in the textbook glossary is incorrect.) The volumes of the spheres give the following sequence:

$$\frac{4}{3}\pi\times 64^3, \frac{4}{3}\pi\times 32^3, \frac{4}{3}\pi\times 16^3, \frac{4}{3}\pi\times 8^3, \frac{4}{3}\pi\times 4^3, \frac{4}{3}\pi\times 2^3, \frac{4}{3}\pi\times 1^3, \ldots$$

The sixth sphere will have a volume of $\frac{4}{3}\pi\times 2^3\ \text{cm}^3$, or $\frac{32\pi}{3}\ \text{cm}^3$. The *n*th sphere will have a volume (in cubic centimetres) of

$$\begin{aligned}\frac{4}{3}\pi\times\left(\frac{64}{2^{n-1}}\right)^3 &= \frac{4}{3}\pi\times\left(\frac{2^6}{2^{n-1}}\right)^3\\ &= \frac{4}{3}\pi\times\left(\frac{2^{18}}{2^{3n-3}}\right)^{3n}\\ &= \frac{4}{3}\pi\times 2^{18-(3n-3)}\\ &= \frac{2^2}{3}\pi\times 2^{21-3n}\\ &= \frac{\pi}{3}\times 2^{23-3n}\end{aligned}$$

c. **Textbook exercise 6 of "Exercises: Extending Your Thinking," p. 291**

6. The following table shows information regarding the first four cubes in the sequence. The answers in the textbook assume that the initial values are not part of the sequence.

Cube	Side Length (cm)	Total Length of Edges (cm)	Surface Area (cm^2)	Volume (cm^3)
1	x	$12x$	$6x^2$	x^3
2	$2x$ $= 2^1 \times x$	$24x$ $= 12 \times 2x$	$24x^2$ $= 6 \times (2x)^2$	$8x^3$ $= (2x)^3$
3	$4x$ $= 2^2 \times x$	$48x$ $= 12 \times 4x$ $= 12 \times 2^2 x$	$96x^2$ $= 6 \times (4x)^2$ $= 6 \times (2^2 x)^2$	$64x^3$ $= (3x)^3$ $= (2^2 x)^3$
4	$8x$ $= 2^3 \times x$	$96x$ $= 12 \times 8x$ $= 12 \times 2^3 x$	$384x^2$ $= 6 \times (8x)^2$ $= 6 \times (2^3 x)^2$	$512x^3$ $= (8x)^3$ $= (2^3 x)^3$

The total length of the edges is obtained by multiplying the number of edges (12) by the length of an edge. The length of the *n*th edge is $2^{n-1} x$. This gives the following expressions, each of which represents the total length of the edges of the *n*th cube.

$$12 \times 2^{n-1} x \quad \text{or} \quad 3 \times 2^2 \times 2^{n-1} x = 3 \times 2^{n+1} x$$

The surface area of the cube is obtained by multiplying the number of faces (6) by the area of one face. The area of one face of the *n*th cube is $\left(2^{n-1} x\right)^2$. This gives the following expressions, each of which represents the surface area of the *n*th cube.

$$6 \times \left(2^{n-1} x\right)^2 \quad \text{or} \quad 6 \times 2^{2n-2} x^2 \quad \text{or} \quad 3 \times 2 \times 2^{2n-2} x^2 = 3 \times 2^{2n-1} x^2$$

The volume of a cube is obtained by cubing the length of an edge. The length of the *n*th edge is $2^{n-1} x$. This gives the following expressions, each of which represents the volume of the *n*th cube.

$$\left(2^{n-1} x\right)^3 \quad \text{or} \quad 2^{3n-3} x^3$$

4. Textbook exercise "Communicating the Ideas," p. 291

Most of the fractals discussed in the textbook have a finite area and an infinite perimeter. The designs increase the perimeter by a factor larger than 1, but increase the area by a factor smaller than 1. The end result is an ever increasing and unbounded perimeter, along with an ever increasing but bounded area.

As you saw in Investigation 1, the perimeter of the Koch snowflake for the *n*th iteration was given by the expression $27\left(\frac{4}{3}\right)^n$. Here the perimeter increases by a factor of $\frac{4}{3}$ at each iteration.

The area of the Koch snowflake also increases, but by a smaller amount at each iteration. The area for the *n*th iteration was given by the expression

$$108+12\times\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}$$

Here the area increases, but only by

$$\begin{aligned}12\times\left(\frac{1-\left(\frac{4}{9}\right)^n}{\frac{5}{9}}-\frac{1-\left(\frac{4}{9}\right)^{n-1}}{\frac{5}{9}}\right)&=12\times\frac{9}{5}\times\left[\left(\frac{4}{9}\right)^{n-1}-\left(\frac{4}{9}\right)^n\right]\\&=12\times\frac{9}{5}\times\left[\left(\frac{4}{9}\right)^{n-1}\left(1-\frac{4}{9}\right)\right]\\&=12\times\frac{\cancel{9}}{\cancel{5}}\times\left[\left(\frac{4}{9}\right)^{n-1}\left(\frac{\cancel{5}}{\cancel{9}}\right)\right]\\&=12\times\left(\frac{4}{9}\right)^{n-1}\end{aligned}$$

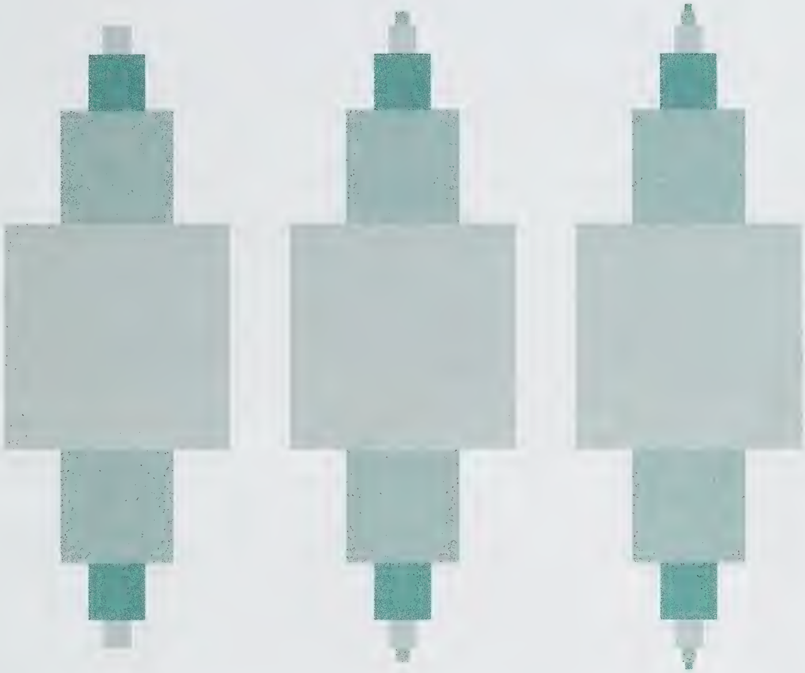
Thus, the area is increasing, but by a smaller amount with each iteration. In fact, the area will never exceed 129.6.

Module Review

1. Textbook exercises 1 and 2 of Part A of "What Should I Be Able to Do?," pp. 295 and 296

1. a. to c. These diagrams are shown in the textbook after exercise 1.e.

d. and e.



2. **a.** Below the values in the table are the calculations used to obtain the values.

Iteration	Length of Side of Newest Square (cm)	Increase in Area of Fractal (cm^2)	Total Area (cm^2)	Increase in Perimeter (cm)	Total Perimeter (cm)
Original	64	NA	4096	NA	256
1	32	2048 $= 2 \times 32^2$	6144 $= 4096 + 2048$	192 $= 6 \times 32$	448 $= 256 + 192$
2	16	512 $= 2 \times 16^2$	6656 $= 6144 + 512$	96 $= 6 \times 16$	544 $= 448 + 96$
3	8	128 $= 2 \times 8^2$	6784 $= 6656 + 128$	48 $= 6 \times 8$	592 $= 544 + 48$
4	4	32 $= 2 \times 4^2$	6816 $= 6784 + 32$	24 $= 6 \times 4$	616 $= 592 + 32$
5	2	8 $= 2 \times 2^2$	6824 $= 6816 + 8$	12 $= 6 \times 2$	628 $= 616 + 12$

b. After iteration 6, the total area will increase by $2\ \text{cm}^2$ to $6826\ \text{cm}^2$. The total perimeter will increase by 6 cm to 634 cm.

c. The total area and total perimeter will continue to increase by smaller and smaller amounts. The total area will approach $6826\frac{2}{3}\ \text{cm}^2$, and the total perimeter will approach 640 cm.

You can see this in the following spreadsheet.

	A	B	C	D	E	F
1	Iteration	Length of Side of Newest Square (cm)	Increase in Area of Fractal (cm²)	Total Area (cm²)	Increase in Perimeter (cm)	Total Perimeter (cm)
2	Original	64		4096		256
3	1	32.0000000	2048.000000	6144.0000000	192.0000000	448.0000000
4	2	16.0000000	512.0000000	6656.0000000	96.0000000	544.0000000
5	3	8.0000000	128.0000000	6784.0000000	48.0000000	592.0000000
6	4	4.0000000	32.0000000	6816.0000000	24.0000000	616.0000000
7	5	2.0000000	8.0000000	6824.0000000	12.0000000	628.0000000
8	6	1.0000000	2.0000000	6826.0000000	6.0000000	634.0000000
9	7	0.5000000	0.5000000	6826.5000000	3.0000000	637.0000000
10	8	0.2500000	0.1250000	6826.6250000	1.5000000	638.5000000
11	9	0.1250000	0.0312500	6826.6562500	0.7500000	639.2500000
12	10	0.0625000	0.0078125	6826.6640625	0.3750000	639.6250000
13	11	0.0312500	0.0019531	6826.6660156	0.1875000	639.8125000
14	12	0.0156250	0.0004883	6826.6665039	0.0937500	639.9062500
15	13	0.0078125	0.0001221	6826.6666260	0.0468750	639.9531250
16	14	0.0039063	0.0000305	6826.6666565	0.0234375	639.9765625
17	15	0.0019531	0.0000076	6826.6666641	0.0117188	639.9882813
18	16	0.0009766	0.0000019	6826.6666660	0.0058594	639.9941406
19	17	0.0004883	0.0000005	6826.6666665	0.0029297	639.9970703
20	18	0.0002441	0.0000001	6826.6666666	0.0014648	639.9985352
21	19	0.0001221	0.0000000	6826.6666667	0.0007324	639.9992676
22	20	0.0000610	0.0000000	6826.6666667	0.0003662	639.9996338

You can also calculate these values in the following manner.

Total Area

The value the total area will approach can be found as follows. Let SUM represent the sum of the areas added in the first n iterations.

$$\begin{aligned} \therefore\ \text{SUM} &= 2\times\left(32^2+16^2+8^2+\ldots+12^{12-2n}\right) \\ &= 2\times\left(2^{10}+2^8+2^6+\ldots+2^{12-2n}\right) \end{aligned}$$

Remember the trick of multiplying by the common ratio, where each term is reduced by a factor of 2^{-2}.

$$\begin{aligned} 2^{-2}\times\text{SUM} &= 2\times\left(2^{-2}\times2^{10}+2^{-2}\times2^8+2^{-2}\times2^6+\ldots+2^{-2}\times2^{12-2n}\right) \\ \frac{\text{SUM}}{4} &= 2\times\left(2^8+2^6+2^4+\ldots+2^{12-2(n+1)}\right) \end{aligned}$$

Subtracting this equation from the earlier one will give a way to find the maximum possible area.

$$\begin{array}{rl} \text{SUM} = 2\times\left(2^{10}+2^8+2^6+\ldots+2^{12-2n}\right) & 1 \\ \dfrac{\text{SUM}}{4} = 2\times\left(2^8+2^6+\ldots+2^{12-2n}+2^{12-2(n+1)}\right) & 2 \\ \hline \dfrac{3}{4}\text{SUM} = 2\times\left(2^{10}+0+0+\ldots+0-2^{12-2(n+1)}\right) & 1-2 \end{array}$$

$$\begin{aligned} \frac{3}{4}\text{SUM} &= 2\times\left(2^{10}-2^{10-2n}\right) \\ \text{SUM} &= \frac{8}{3}\times\left[2^{10}\left(1-2^{-2n}\right)\right] \\ &= \frac{8192}{3}\left(1-2^{-2n}\right) \end{aligned}$$

SUM is actually 1 SUM, and $\frac{\text{SUM}}{4}$ is $\frac{1}{4}$ SUM. So, $\text{SUM}-\frac{\text{SUM}}{4}=\frac{3}{4}\text{SUM}$.

As n increases, the value of 2^{-2n} decreases and approaches 0 quickly. This means the total area approaches $4096+\frac{8192}{3}=6826\frac{2}{3}$. **Note:** 4096 is the area of the initial square.

Total Perimeter

The value the total perimeter will approach can be found as follows. Let SUM represent the sum of the perimeters added in the first n iterations.

$$\begin{aligned} \text{SUM} &= 6\times\left(32+16+8+\ldots+2^{6-n}\right) \\ &= 6\times\left(2^5+2^4+2^3+\ldots+2^{6-n}\right) \end{aligned}$$

Remember the trick of multiplying by the common ratio, where each term is reduced by $\frac{1}{2}$ or 2^{-1}.

$$2^{-1} \times \text{SUM} = 6 \times \left(2^{-1} \times 2^5 + 2^{-1} \times 2^4 + 2^{-1} \times 2^3 + \ldots + 2^{-1} \times 2^{6-n}\right)$$

$$\frac{\text{SUM}}{2} = 6 \times \left(2^4 + 2^3 + 2^2 + \ldots + 2^{6-(n+1)}\right)$$

Subtracting this equation from the earlier one will give a way to find the maximum possible perimeter.

$$\text{SUM} = 6 \times \left(2^5 + 2^4 + 2^3 + \ldots + 2^{6-n}\right) \quad (1)$$

$$\frac{\text{SUM}}{2} = 6 \times \left(2^4 + 2^3 + \ldots + 2^{6-n} + 2^{6-(n+1)}\right) \quad (2)$$

$$\frac{\text{SUM}}{2} = 6 \times \left(2^5 + 0 + 0 + \ldots + 0 - 2^{6-(n+1)}\right) \quad (1) - (2)$$

$$\frac{\text{SUM}}{2} = 6 \times \left(2^5 - 2^{5-n}\right)$$

$$\text{SUM} = 12 \times \left[2^5\left(1 - 2^{-n}\right)\right]$$

As n increases, the value of 2^{-n} decreases and approaches 0 quickly. That means the total perimeter approaches $256 + 12 \times 2^5 = 640$. **Note:** 256 is the perimeter of the initial square.

2. **Textbook exercises 4, 6, and 8 of Part B of "What Should I Be Able to Do?," pp. 296 to 298**

   4. a. The following chart shows the approximate amounts of Amoxicillin remaining in the child's body after each two-hour interval.

Time	Number of Hours	Amount Remaining (mg)
7:00 A.M. or 07:00	0	250
9:00 A.M. or 09:00	2	125
11:00 A.M. or 11:00	4	62.5
1:00 P.M. or 13:00	6	31.25
3:00 P.M. or 15:00	8	15.625

b. From the table, 11:45 is in the interval from 4 h to 6 h with between 62.5 mg and 31.25 mg remaining.

$62.5 - 31.25 = 31.25$ (approximate amount of Amoxicillin decrease from 11:00 A.M. to 1:00 P.M.).

$\frac{31.25}{2} = 15.625$ (approximate amount of Amoxicillin decrease from 11:00 A.M. to 12:00 noon).

$\frac{3}{4} \times 15.625 = 11.718\ 75$ (approximate amount of Amoxicillin decrease from 11:00 A.M. to 11:45 A.M.).

$62.5 - 11.718\ 75 = 50.781\ 25$ (approximate amount of Amoxicillin remaining at 11:45 A.M.).

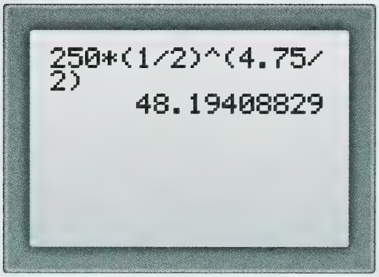
The estimated amount of Amoxicillin remaining at 11:45 A.M. is 50.78 mg (rounded to 2 decimal places). **Note:** This estimated amount is not very accurate as more Amoxicillin will be reduced in the first hour than in the second hour of the two-hour interval.

You can determine the amount of Amoxicillin by using an equation.

The equation $y = 250\left(\frac{1}{2}\right)^{\frac{t}{2}}$ describes the amount of Amoxicillin remaining in the child's body at time t (in hours).

The amount of Amoxicillin remaining at 11:45 A.M. can be calculated from the equation by substituting 4.75 for t.

$$\begin{aligned} y &= 250\left(\frac{1}{2}\right)^{\frac{t}{2}} \\ &= 250\left(\frac{1}{2}\right)^{\frac{4.75}{2}} \\ &\doteq 48.19 \end{aligned}$$



The amount of Amoxicillin in the child's body at 11:45 A.M. is about 48.19 mg.

6. a. The following spreadsheet shows the numbers of deer for the ten years following the initial count. **Note:** You must make sure you round decimal values to the nearest whole number because you cannot have part of a deer.

	A	B	C	D	E	F
1	Year	Population at Start of Year	Number of Human-Related Deaths	Number of Natural Deaths	Number of Births	Population at End of Year
2	1	2200	183	110	383	2290
3	2	2290	183	115	383	2375
4	3	2375	183	119	383	2456
5	4	2456	183	123	383	2533
6	5	2533	183	127	383	2606
7	6	2606	183	130	383	2676
8	7	2676	183	134	383	2742
9	8	2742	183	137	383	2805
10	9	2805	183	140	383	2865
11	10	2865	183	143	383	2922
12	11	2922	183	146	383	2976
13	12	2976	183	149	383	3027

There will be approximately 2922 deer in the region ten years after the initial count.

b. If the pattern continues, the number of deer will exceed 3000 twelve years after the initial count.

At this point if nothing is done, the number of deer dying of natural causes will increase in the following years since the carrying capacity of the area will have been exceeded. This would be a waste of a valuable natural resource.

8. a. The following table shows the dimensions of the six boxes. The values have been rounded to the nearest thousandth where appropriate. **Note:** There are four sides that measure 10 cm by 16 cm and one side that measures 10 cm by 10 cm.

Box Number	Length (cm)	Width (cm)	Height (cm)	Needed Material (cm^2)
1 (original)	10	10	16	740 $= 4(10 \times 16) + 10^2$
2	8	8	12.8	473.6 $= 4(8 \times 12.8) + 8^2$
3	6.4	6.4	10.24	303.104 $= 4(6.4 \times 10.24) + 6.4^2$
4	5.12	5.12	8.192	193.987 $= 4(5.12 \times 8.192) + 5.12^2$
5	4.096	4.096	6.5536	124.151 $= 4(4.096 \times 6.5536) + 4.096^2$
6	3.2768	3.2768	5.242 88	79.457 $= 4(3.2768 \times 5.242\ 88) + 3.2768^2$

$5.242\ 88 \times 3.2768^2 = 56.294\ 995\ 34$

The volume of the sixth box will be about $56.3\ \text{cm}^3$.

b. Total amount of material = Total surface area + Wasted material (10%)

$$
\begin{aligned}
&= (740 + 473.6 + 303.104 + 193.987 + 124.151 + 79.457)^2 \times 1.10 \\
&= 1914.299 \times 1.10 \\
&= 2105.729
\end{aligned}
$$

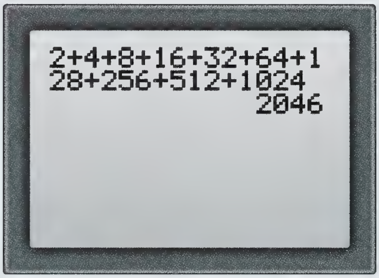
The total amount of material needed is about $2105.729\ \text{cm}^2$.

Enrichment

1. The initial term, a, is 2, and the common ratio, r, is 2.

Substitute these values into the formula to find the sum of the first ten terms. For ten terms, the value of n is 9, since the formula gives the sum of $n+1=10$ terms.

$$\begin{aligned}\text{SUM} &= \frac{a\left(1-r^{n+1}\right)}{1-r}\\ &= \frac{2\left(1-2^{10}\right)}{1-2}\\ &= \frac{2(1-1024)}{-1}\\ &= 2046\end{aligned}$$



2. The initial term, a, is 8, and the common ratio, r, is 0.5.

Substitute these values into the formula to find the sum of the whole sequence. This is true because as n becomes very large, the value of r^n becomes indistinguishable from 0. The formula will let you find the sum of all of the terms if you make r^{n+1} equal to 0.

$$\begin{aligned}\text{SUM} &= \frac{a\left(1-r^{n+1}\right)}{1-r}\\ &= \frac{8\left(1-0.5^{n+1}\right)}{1-0.5}\\ &= \frac{8(1-0)}{0.5}\\ &= 16\end{aligned}$$

3. The initial term, a, is 4, and the common ratio, r, is 3.

Substitute these values into the formula to find the sum of the first 25 terms. For 25 terms, the value of n is 24, since the formula gives the sum of $n+1=25$ terms.

$$\begin{aligned}\text{SUM} &= \frac{a\left(1-r^{n+1}\right)}{1-r}\\ &= \frac{4\left(1-3^{25}\right)}{1-3}\\ &= \frac{4\left(1-531\,441\right)}{-2}\\ &= 1\,062\,880\end{aligned}$$

4. The initial term, a, is $\frac{7}{10}$, and the common ratio, r, is $\frac{1}{10}$.

Substitute these values into the formula to find the sum of the whole sequence. This is true because as n becomes very large, the value of r^n becomes indistinguishable from 0. The formula will let you find the sum of all of the terms if you make r^n equal to 0.

$$\begin{aligned}\text{SUM} &= \frac{a\left(1-r^{n+1}\right)}{1-r}\\ &= \frac{\frac{7}{10}\left[1-\left(\frac{1}{10}\right)^{n+1}\right]}{1-\frac{1}{10}}\\ &= \frac{\frac{7}{10}(1-0)}{\frac{9}{10}}\\ &= \frac{7}{\cancel{10}}\times\frac{\cancel{10}}{9}\\ &= \frac{7}{9}\end{aligned}$$

Completing the Project

Textbook exercises 9 and 10 of Part C of "What Should I Be Able to Do?", pp. 299 and 300

9. You can create the data table in a couple of ways: by time of day or by elapsed time. These are both shown in the following table. Making this table on a spreadsheet would probably be the easiest.

Elapsed Time (h)	Time of Day	Amount of Caffeine (mg)
0	8:00 A.M.	330
1	9:00 A.M.	287.1
2	10:00 A.M.	249.777
3	11:00 A.M.	217.305 99
4	12:00 P.M.	189.056 211 3
5	1:00 P.M.	164.478 903 8
6	2:00 P.M.	143.096 646 3
7	3:00 P.M.	124.494 082 3
8	4:00 P.M.	108.309 851 6
9	5:00 P.M.	94.229 570 9
10	6:00 P.M.	81.979 726 68
11	7:00 P.M.	71.322 362 21
12	8:00 P.M.	62.050 455 13

It is easier to create an exponential equation to describe this data using the Elapsed Time method. The equation is $y = 330 \times (0.87)^t$, where t is the number of hours since 8:00 A.M. and y is the amount of caffeine left in the body (in milligrams). It is easier because the Amount of Caffeine column describes a geometric series with the first term 330 and common ratio of 0.87. If you want an equation that uses the time of day rather than elapsed time, the equation $y = 330 \times (0.87)^{t-8}$ can be used. The term $t - 8$ corrects for the eight-hour offset into the day before the caffeine is taken.

It is also easy to create an exponential regression equation using the data and the TI-83. Simply enter the data into lists L1 and L2 and do an exponential regression. The result will be very similar to the student results on page 299 of the textbook. However, the student has a different equation in part c. and part f. The equation in part f. is correct for the sequence using time of day and amount of caffeine.

In part d., the student has made a rounding error. The answer should be 267.79 mg of caffeine remain in Mr. Jones's body after 1.5 h.

The graphs on the TI-83 make finding the half-life easy. Just add a second graph $(y = 165)$ and find the intersection. You can convert the x-value into hours and minutes by subtracting 8 if you used time of day in your list L1 or read the x-value from your calculator if you used elapsed time as L1. Again, a rounding error was made in the student's answer, with the half-life rounding to 5.0 h.

The student used an equation to determine the answer to part f. (part e. on page 275). You can also use Table feature on your graphing calculator. The student has not numbered the answers the same as in the textbook. The student uses "e." for "d." and "g." for "f." The answer to "g." is incorrect. Twelve hours from 10:00 P.M. is 10:00 A.M., not 8:00 A.M.

10. The findings are justified. Modifying the table on a spreadsheet would probably be the easiest way to build a new data table.

Image Credits

Cover and Title Page
collage: PhotoDisc Collection/Getty Images

Welcome Page
PhotoDisc Collection/Getty Images

Page

2 **collage:** Image Club StudioGear/EyeWire Collection/Getty Images (top and right), PhotoDisc Collection/Getty Images (bottom left)
4 PhotoDisc Collection/Getty Images
7 EyeWire Collection/Getty Images
8–9 **collage:** PhotoDisc Collection/Getty Images
10 **collage:** PhotoDisc Collection/Getty Images (coffee beans and coffee cup)
11 **top:** PhotoDisc Collection/Getty Images
bottom: PhotoDisc Collection/Getty Images
12 PhotoDisc Collection/Getty Images
13 Image Club StudioGear/EyeWire Collection/Getty Images
16 Image Club StudioGear/EyeWire Collection/Getty Images
17 Image Club StudioGear/EyeWire Collection/Getty Images
18 **top left:** Corel Corporation
top right: © 2002 www.clipart.com
19 Rubberball Productions/Getty Images
20 **top left:** PhotoDisc Collection/Getty Images
top right: © 2002 www.clipart.com
bottom: Corel Corporation
21 PhotoDisc Collection/Getty Images
26 PhotoDisc Collection/Getty Images
31 **bottom:** PhotoDisc Collection/Getty Images
32 **bottom:** PhotoDisc Collection/Getty Images
36 Image Club StudioGear/EyeWire Collection/Getty Images
38 PhotoDisc Collection/Getty Images
40 **collage:** PhotoDisc Collection/Getty Images (coffee beans and coffee cup)
41 EyeWire Collection/Getty Images
42 PhotoDisc Collection/Getty Images
43 **collage:** PhotoDisc Collection/Getty Images

ILL